# DUAL DECOMPOSITION OF MINIMAL COUNTEREXAMPLES

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Abstract.

## 1. Introduction

By Tait's theorem, a plane triangulation $G$ is properly 4-vertex-colourable if and only if its dual cubic graph $G'$ is properly 3-edge-colourable. We propose to obstruct a minimal counterexample to the Four Colour Theorem by decomposing its dual along a minimum edge cut, 3-edge-colouring the two resulting pieces, and recombining the colourings.

## 2. Strategy

The argument proceeds in six steps.

(1) **Minimal counterexample.** Assume a smallest maximal planar graph $G$ admitting no proper 4-colouring. Standard reductions let us take $G$ to be 5-connected, so its dual $G'$ is a cyclically 5-edge-connected cubic plane graph.
(2) **Dualise.** Pass to the dual cubic plane graph $G'$, where, by Tait, 4-colourability of $G$ is equivalent to 3-edge-colourability of $G'$.
(3) **Minimum edge cut.** Take a smallest edge cut of $G'$ separating it into two components $A$ and $B$. (The size of the smallest cyclic edge cut equals the cyclic edge connectivity of $G'$; record which cut is used and why it is minimal.)
(4) **Cap to cubic.** Add the minimum number of edges (and, if needed, vertices) to each of $A$ and $B$ to absorb the severed cut edges, yielding two cubic plane graphs $A^+$ and $B^+$, each smaller than $G'$.
(5) **Colour the pieces.** Since $A^+$ and $B^+$ are smaller than the minimal counterexample's dual, each admits a proper 3-edge-colouring.
(6) **Reconnect.** Prove that the two 3-edge-colourings can be made compatible across the cut — after a colour permutation and Kempe-type adjustments — so that they glue to a proper 3-edge-colouring of $G'$, contradicting the assumption that $G$ is a counterexample.

*Remark* 2.1. Steps (1)–(5) are assembled from standard machinery (minimality, Tait duality, edge-cut/girth duality, and the inductive existence of colourings on smaller cubic graphs). The load-bearing claim is step (6): the compatibility of the two boundary colourings across the cut. The parity of colours along an edge cut

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in a 3-edge-colouring is constrained, and the goal is to show these constraints can always be satisfied simultaneously on both sides.

## 3. Step 1: The minimal counterexample

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree and connectivity properties a smallest counterexample must have.

**Lemma 3.1** (Reduction to triangulations)**.** *If every triangulation is properly 4-vertex-colourable, then so is every plane graph.*

*Proof.* Let $H$ be a plane graph. Add edges to $H$, maintaining planarity, until no further edge can be added; the result is a triangulation $H^+$ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of $H^+$ restricts to a proper 4-colouring of $H$, since every edge of $H$ is an edge of $H^+$. □

By Lemma 3.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

**Definition 3.2** (Minimal counterexample)**.** Let $G$ be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call $G$ a *minimal counterexample.* By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

*Remark* 3.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being $K_4$), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

**Lemma 3.4** (Minimum degree)**.** *A minimal counterexample $G$ has minimum degree $\delta(G) \geq 5$.*

*Proof.* Suppose some vertex $v$ has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then $G'$ is a plane graph on fewer vertices, so by Definition 3.2 and Lemma 3.1 it has a proper 4-colouring. The at most three neighbours of $v$ use at most three colours, so a fourth colour is free for $v$, extending the colouring to $G$ — a contradiction.

If $d = 4$, again 4-colour $G - v$. If the four neighbours of $v$ use at most three colours we extend as before, so assume they receive all four colours; let $v_1, v_2, v_3, v_4$ be the neighbours in cyclic order around $v$, coloured $1, 2, 3, 4$. Consider the subgraph induced by the colour classes 1 and 3, and let $K$ be its connected component containing $v_1$. If $v_3 \notin K$, swap colours 1 and 3 on $K$; now no neighbour of $v$ is coloured 1, freeing it for $v$. If $v_3 \in K$, then a 1–3 Kempe chain joins $v_1$ to $v_3$, and this chain together with $v$ encloses exactly one of $v_2, v_4$; hence the 2–4 component containing $v_2$ cannot also reach $v_4$, and swapping colours 2 and 4 on it frees colour 2 for $v$. Either way the colouring extends to $G$, a contradiction.

Hence $\delta(G) \geq 5$. □

For the connectivity reduction we recall that, in a triangulation, the relevant small separators are short cycles.

**Definition 3.5** (Separating cycle)**.** A cycle $C$ in a triangulation $G$ is *separating* if it is not a face boundary; equivalently, both the open interior and the open exterior of $C$ (with respect to the embedding) contain at least one vertex of $G$.

**Lemma 3.6** (No separating triangle)**.** *A minimal counterexample $G$ has no separating triangle; consequently $G$ is* 4*-connected.*

*Proof.* Suppose $T = xyz$ is a separating triangle. It splits $G$ into the triangulation $G_{\text{in}}$ induced by $T$ and the vertices interior to $T$, and the triangulation $G_{\text{out}}$ induced by $T$ and the vertices exterior to $T$; in each, $T$ bounds a face. Both are triangulations on fewer vertices than $G$, so by minimality each has a proper 4-colouring. Permuting colours in $G_{\text{out}}$, we may assume the two colourings agree on $\{x, y, z\}$ (the three corners receive distinct colours in each, so a colour permutation aligns them). The two colourings then glue to a proper 4-colouring of $G$, a contradiction. A triangulation with no separating triangle and $\delta \geq 5$ is 4-connected, since a minimal vertex cut of size $\leq 3$ in a triangulation is the vertex set of a separating cycle of that length, and a 3-cut would give a separating triangle. □

**Proposition 3.7** (5-connectivity)**.** *A minimal counterexample $G$ may be taken to be* 5*-connected; equivalently, $G$ has no separating triangle and no separating* 4*-cycle.*

*Discussion.* By Lemma 3.6 it remains to eliminate separating 4-cycles. This is the classical reducibility step of Birkhoff: a shortest separating 4-cycle $C = wxyz$ bounds inner and outer triangulations; one 4-colours both by minimality and reconciles their colourings on $C$ using Kempe-chain exchanges, the only obstruction being the two ways the four corners can be coloured with three or four colours. Carrying out the exchanges shows the colourings can always be aligned, so $G$ has no separating 4-cycle. As with Lemma 3.6, the absence of separating 3- and 4-cycles in a triangulation with $\delta \geq 5$ is equivalent to 5-connectivity. We take this reduction as standard and refer to the literature on the Four Colour Theorem for the detailed Kempe-chain bookkeeping. □

The output of this step — a 5-connected triangulation $G$ with $\delta(G) \geq 5$ and no separating triangle or 4-cycle — is exactly the hypothesis the dual construction of Step 2 consumes.

## 4. Step 2: The dual cubic graph

## 5. Step 3: The minimum edge cut

## 6. Step 4: Capping to cubic planar graphs

## 7. Step 5: Colouring the pieces

## 8. Step 6: Reconnecting the colourings