FACE-MONOCHROMATIC PAIRS AND THE FOUR COLOUR THEOREM

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ABSTRACT. We propose the *face-monochromatic-pair conjecture*, a structural property of proper 3-edge-colourings of cubic plane graphs that, if true, implies the Four Colour Theorem. Working in the planar dual G' of a hypothetical minimal counterexample G to 4CT, we delete a single pentagonal face of G' and rewire its five external vertices around a new apex vertex and a chord; the resulting *reduced dual* $\widehat{G}'_{v,i}$ is a smaller cubic plane graph whose proper 3-edge-colourings, by the minimality of G, are constrained by a chord-apex condition and a pair of Kempe-cycle conditions. The face-monochromatic-pair conjecture, in its strengthened form, asserts the existence in every such colouring of a face F and two non-incident same-coloured edges $e_1, e_2 \in \partial F$ whose subdivision-and-bridging produces a 4-face f_n whose boundary colouring places it under the hypothesis of a 4-face edge-suppression theorem; we use this theorem to derive a proper 3-edge-colouring of G', contradicting minimality. We verify the conjecture computationally on all chord-apex+Kempe colourings of reduced duals with $|V(G)| \leq 20$ (142,812 colourings, all pass); the weaker form is verified up to $|V(G)| \leq 21$ (535,182 colourings, all pass).

1. INTRODUCTION

The Four Colour Theorem (4CT) — that every loopless plane graph admits a proper 4-vertex-colouring — has, since the late 1970s, only been proved by computer-assisted case analysis on a discharging argument over a finite set of unavoidable reducible configurations. Appel and Haken's original proof [2, 3], the Robertson–Sanders–Seymour–Thomas reworking [4], and Gonthier's machine-checked version [5] all share that structure.

This paper takes a different approach: rather than discharge over configurations in the triangulation G, we work in its planar dual G', a cubic plane graph whose proper 3-edge-colourings correspond by Tait's theorem to proper 4-vertex-colourings of G. Assuming G is a minimal counterexample to 4CT, we delete a single pentagonal face of G' and rewire its five external vertices, obtaining a smaller cubic plane graph $\widehat{G}'_{v,i}$ — the *reduced dual* — which by minimality *is* properly 3-edge-colourable. Two structural lemmas constrain every such colouring: a *chord-apex* condition (Lemma 3.7) forcing two named edges to share a colour, and a pair of *Kempe-cycle* conditions (Lemma 3.8) placing four of the rewired edges on common bichromatic Kempe cycles. These constraints are the starting point of the present development.

The main contribution of the paper is the *face-monochromatic-pair conjecture* (Conjecture 5.1) and its strengthening (Conjecture 5.22), which we show together

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imply the Four Colour Theorem. The supporting ingredients are the chord-apex and Kempe-cycle lemmas on reduced-dual colourings, the classical operation of *edge suppression* (delete the edge and smooth its two degree-2 endpoints; equivalently, simple-graph contraction in the dual triangulation; recalled in Section 4), and an observation that suppression preserves 3-edge-colourability when applied across a 4-face whose two opposite boundary edges carry different colours (Theorem 4.2).

The strategy is to construct, from a putative minimal counterexample's reduced-dual colouring, a 4-face f_n in a slightly modified graph $\widehat{G}'^{+}$ to which the suppression theorem applies; the suppression then produces a properly 3-edge-coloured graph from which a 3-edge-colouring of G' can be recovered, contradicting the non-4-colourability of G. The face-monochromatic-pair conjecture asserts the existence of the structural data (F, e_1, e_2) needed to build f_n; the strengthening guarantees that f_n's boundary colouring falls under the suppression theorem's hypothesis. Both conjectures have been verified computationally on all chord-apex+Kempe colourings of reduced duals up to $|V(G)| \leq 20$, with the weaker form going up to $|V(G)| \leq 21$.

Organization. Section 2 fixes the minimal-counterexample framework: G is a triangulation, $\delta(G) \geq 5$, and every triangulation on fewer vertices is properly 4-colourable. Section 3 introduces the reduced dual $\widehat{G}'_{v,i}$ and proves the chord-apex and Kempe-cycle lemmas. Section 4 defines edge suppression and proves its 4-face 3-edge-colourability theorem. Section 5 states the two conjectures, reports the empirical verification, and gives the implication to 4CT.

Companion paper. An iterated version of the reduced-dual construction — producing a sequence $H_1, H_2, \ldots$ of progressively smaller cubic plane graphs and tracking an accumulating "protected" edge set — is the subject of a companion paper. The present paper does not use that iteration.

2. The minimal counterexample

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

Lemma 2.1 (Reduction to triangulations)**.** *If every triangulation is properly* 4*-vertex-colourable, then so is every plane graph.*

Proof. Let H be a plane graph. Add edges to H, maintaining planarity, until no further edge can be added; the result is a triangulation H^+ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of H^+ restricts to a proper 4-colouring of H, since every edge of H is an edge of H^+. □

By Lemma 2.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

Definition 2.2 (Minimal counterexample)**.** Let G be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call G a *minimal counterexample.* By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

Remark 2.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being K_4), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

Lemma 2.4 (Minimum degree)**.** *A minimal counterexample G has minimum degree $\delta(G) \geq 5$.*

Proof. Suppose some vertex v has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then G' is a plane graph on fewer vertices, so by Definition 2.2 and Lemma 2.1 it has a proper 4-colouring. The at most three neighbours of v use at most three colours, so a fourth colour is free for v, extending the colouring to G — a contradiction.

If $d = 4$, again 4-colour $G-v$. If the four neighbours of v use at most three colours we extend as before, so assume they receive all four colours; let v_1, v_2, v_3, v_4 be the neighbours in cyclic order around v, coloured $1, 2, 3, 4$. Consider the subgraph induced by the colour classes 1 and 3, and let K be its connected component containing v_1. If $v_3 \notin K$, swap colours 1 and 3 on K; now no neighbour of v is coloured 1, freeing it for v. If $v_3 \in K$, then a 1–3 Kempe chain joins v_1 to v_3, and this chain together with v encloses exactly one of v_2, v_4; hence the 2–4 component containing v_2 cannot also reach v_4, and swapping colours 2 and 4 on it frees colour 2 for v. Either way the colouring extends to G, a contradiction.

Hence $\delta(G) \geq 5$. □

3. The reduced dual

Write G' for the dual of G: since G is a triangulation, G' is a cubic plane graph in which each vertex of G corresponds to a face of G', each face of G to a vertex of G', and each edge to a dual edge. A vertex of G of degree k corresponds to a k-gonal face of G'.

The following labelling of vertices in a properly 3-edge-coloured cubic plane graph will be useful in Section 5.

Definition 3.1 (Heawood number of a vertex)**.** Let H be a cubic plane graph with a fixed planar embedding, and let $\varphi\colon E(H) \to \{1, 2, 3\}$ be a proper 3-edge-colouring. At each vertex $v \in V(H)$, the three incident edges receive three distinct colours; reading them in clockwise order around v gives a cyclic permutation of $(1, 2, 3)$. The *Heawood number* of v is

$$h_\varphi(v) := \begin{cases} +1 & \text{if the clockwise cyclic colour order at } v \text{ is } (1,2,3), \\ -1 & \text{if it is } (1,3,2). \end{cases}$$

Equivalently, $h_\varphi(v) = +1$ when the clockwise colour order at v is an even cyclic permutation of $(1, 2, 3)$ and -1 when it is an odd one. The labels are due to Heawood [1], who introduced them as part of his analysis of 3-edge-colourings of cubic plane graphs.

By Lemma 2.4, $\delta(G) \geq 5$, and Euler's formula gives $\sum_{u \in V(G)}(6 - \deg u) = 12$, so G has a vertex of degree exactly 5 (indeed at least twelve). Fix such a vertex v. Its dual face F_v is a pentagon, bounded by the five dual vertices corresponding to the five faces of G incident to v.

Definition 3.2 (Reduced dual)**.** Let v be a degree-5 vertex of G with pentagonal dual face F_v, and fix an index $i \in \{0, 1, 2, 3, 4\}$. The *reduced dual* $\widehat{G}'_{v,i}$ is the plane graph obtained from G' as follows.

(1) Delete the five dual vertices on the boundary of F_v, together with all edges incident to them. Each deleted vertex is cubic, with two edges on ∂F_v and one edge leaving F_v; deleting the five boundary vertices therefore removes the five external edges as well, dropping their five outer endpoints from degree 3 to degree 2. These five degree-2 vertices lie on the boundary of a single face F of the resulting graph.
(2) List the five degree-2 vertices in clockwise order around F as $A = (A_0, A_1, A_2, A_3, A_4)$.
(3) Add a new vertex v_n and join it to A_i, A_{i+1}, and A_{i+2} (indices mod 5) by three new edges.
(4) Add a new edge between A_{i+3} and A_{i+4} (indices mod 5).

Remark 3.3. Steps (3) and (4) restore cubicity: A_i, A_{i+1}, A_{i+2} each gain one edge to v_n and A_{i+3}, A_{i+4} each gain the new edge, so all five return to degree 3, and v_n has degree 3. Since $A_i, \dots, A_{i+2}$ and A_{i+3}, A_{i+4} are each consecutive along ∂F, the new vertex and edge can be drawn inside F without crossings, so $\widehat{G}'_{v,i}$ is again a cubic plane graph. The construction depends on the choice of i up to the rotational symmetry of A.

Definition 3.4 (Edges of the reduced dual)**.** The four edges added in steps (3) and (4) of Definition 3.2 are named as follows. The chord $A_{i+3}A_{i+4}$ is the *merged edge*; the edge $A_{i+1}v_n$ is the *spike edge*; the edge A_iv_n is the *side*-0 *edge*; and the edge $A_{i+2}v_n$ is the *side*-1 *edge*. In the $i = 0$ case of Figure 1 these are $\{A_3, A_4\}$, $\{A_1, v_n\}$, $\{A_0, v_n\}$, and $\{A_2, v_n\}$ respectively.

We will use the following structural fact about proper 3-edge-colourings near a pentagonal face of a cubic plane graph; it is stated for a generic such graph H, not specifically for the reduced dual.

Lemma 3.5 (Pentagonal externals)**.** *Let H be a cubic plane graph and F a pentagonal face of H, with ∂F traversed clockwise as u_0, u_1, u_2, u_3, u_4. For each i let f_i be the unique edge of H incident to u_i that does not lie on ∂F. An assignment φ of colours from $\{1, 2, 3\}$ to the ten edges incident to $\{u_0, \dots, u_4\}$ is proper at every u_i if and only if there is some index j such that*

$$\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2}) \quad \textit{and} \quad \{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1, 2, 3\} \setminus \{\varphi(f_j)\},$$

indices mod 5*.*

Proof. Write $e_i = u_iu_{i+1}$ for the boundary edges of ∂F (indices mod 5). A colouring φ is proper at every u_i if and only if at each u_i the three incident edges e_{i-1}, e_i, f_i receive three distinct colours; whenever this holds, $\varphi(f_i)$ is forced to be the unique colour in $\{1, 2, 3\} \setminus \{\varphi(e_{i-1}), \varphi(e_i)\}$, and φ restricts to a proper 3-edge-colouring of the cycle ∂F.

(⇒) The line graph of ∂F is C_5, whose maximum independent set has size 2, so no colour appears more than twice on ∂F; and since ∂F is an odd cycle, all three colours appear. The colour multiset on $(\varphi(e_0), \dots, \varphi(e_4))$ is therefore $(2, 2, 1)$, with the singleton at a unique position. Cyclically shifting indices we may place this position at 0; let c be the singleton colour. The remaining four edges form the path

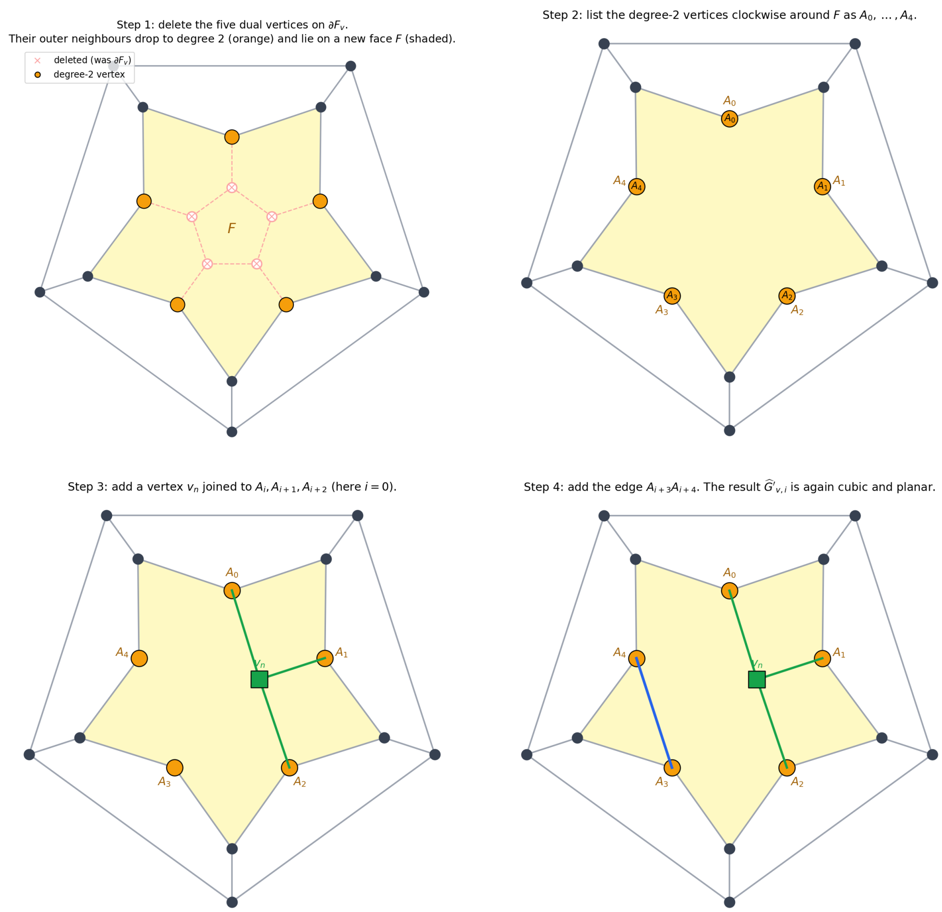

FIGURE 1. The four steps of Definition 3.2, illustrated on G' = the dodecahedron (dual of the icosahedron) with F_v the inner pentagon and $i = 0$. Top left: delete the five boundary vertices of F_v, leaving five degree-2 vertices on a new face F. Top right: order them clockwise as $A_0, \dots, A_4$. Bottom left: add v_n joined to A_0, A_1, A_2. Bottom right: add the chord A_3A_4, giving the cubic plane graph $\widehat{G}'_{v,0}$.

$e_1e_2e_3e_4$, which by propriety alternates between the other two colours, so for some labelling $\{a, b, c\} = \{1, 2, 3\}$,

$$(\varphi(e_0), \varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)) = (c, a, b, a, b).$$

Reading off the forced values of $\varphi(f_i)$,

$$\varphi(f_0) = a, \quad \varphi(f_1) = b, \quad \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c,$$

which is the lemma's pattern at $j = 2$ (the cyclic shift maps this back to the corresponding j in the original indexing). This case is the unique proper 3-edge-colouring of ∂F up to cyclic shift and permutation of $\{1, 2, 3\}$ (since $5 \cdot 3! = 30 = P(C_5, 3)$, the chromatic polynomial of C_5 at 3), so it exhausts every proper φ.

($\Leftarrow$) The lemma's hypothesis is invariant under cyclic shifts of indices and under permutations of $\{1,2,3\}$, so we may assume $j=2$, $\varphi(f_2)=\varphi(f_3)=\varphi(f_4)=c$, $\varphi(f_0)=a$, and $\varphi(f_1)=b$, with $\{a,b,c\}=\{1,2,3\}$. Propriety at u_i and u_{i+1} requires $\varphi(e_i)\notin\{\varphi(f_i),\varphi(f_{i+1})\}$, which gives

$$\varphi(e_0)=c, \quad \varphi(e_1)=a, \quad \varphi(e_2)\in\{a,b\}, \quad \varphi(e_3)\in\{a,b\}, \quad \varphi(e_4)=b.$$

The remaining propriety condition $\varphi(e_{i-1})\neq\varphi(e_i)$ holds automatically at u_0,u_1,u_4, forces $\varphi(e_2)=b$ at u_2, and then forces $\varphi(e_3)=a$ at u_3. The resulting triples $(\varphi(e_{i-1}),\varphi(e_i),\varphi(f_i))$ at u_0,u_1,u_2,u_3,u_4 are

$$(b,c,a), \quad (c,a,b), \quad (a,b,c), \quad (b,a,c), \quad (a,b,c),$$

each a permutation of $\{1,2,3\}$, so φ is proper at every u_i. $\square$

Remark 3.6. The two-element condition $\{\varphi(f_{j+3}),\varphi(f_{j+4})\}=\{1,2,3\}\setminus\{\varphi(f_j)\}$ cannot be dropped: a 3-colouring satisfying $\varphi(f_j)=\varphi(f_{j+1})=\varphi(f_{j+2})$ alone need not extend, e.g. $(1,1,1,1,2)$.

Since $\widehat{G}'_{v,i}$ is the dual of a triangulation on fewer vertices than G, it is 3-edge-colourable by the minimality of G. The following lemma constrains every such colouring.

Lemma 3.7. *Let G be a minimal counterexample, and let $\widehat{G}'_{v,i}$ be a reduced dual of its dual G'. Then in every proper 3-edge-colouring of $\widehat{G}'_{v,i}$, the merged edge and the spike edge receive the same colour.*

Proof. After cyclically relabelling, assume $i=0$. Suppose for contradiction that φ is a proper 3-edge-colouring of $\widehat{G}'_{v,0}$ in which the merged edge $\{A_3,A_4\}$ and the spike edge $\{A_1,v_n\}$ receive different colours (Figure 2, top), and write

$$X=\varphi(\{A_0,v_n\}), \quad Y=\varphi(\{A_1,v_n\}), \quad Z=\varphi(\{A_2,v_n\}), \quad W=\varphi(\{A_3,A_4\}).$$

Propriety of φ at v_n forces $\{X,Y,Z\}=\{1,2,3\}$, and the assumption $W\neq Y$ leaves $W\in\{X,Z\}$.

We lift φ to a colouring ψ of $E(G')$ as follows. Let $B_0,\dots,B_4$ be the boundary vertices of ∂F_v in G', indexed so that $f_k=B_kA_k$. On every edge of G' that survived the reduction, set $\psi=\varphi$. At each A_k the two surviving edges retain their φ-colours, so the remaining edge at A_k — in G' this is the external f_k; in $\widehat{G}'_{v,0}$ this is a v_n-edge ($k\in\{0,1,2\}$) or the chord ($k\in\{3,4\}$) — is forced to take the third colour at A_k. Since the two-surviving-edge colours at A_k agree in G' and $\widehat{G}'_{v,0}$, the third colour does too, giving

$$\psi(f_0)=X, \quad \psi(f_1)=Y, \quad \psi(f_2)=Z, \quad \psi(f_3)=\psi(f_4)=W$$

(the last two equalities holding because the chord is a single edge contributing its colour at each of A_3 and A_4).

It remains to assign colours to the five boundary edges B_kB_{k+1} of ∂F_v. Apply Lemma 3.5 to G' at the face F_v with the B_k's as its boundary vertices and the same indexing. The external vector $(\psi(f_0),\dots,\psi(f_4))=(X,Y,Z,W,W)$ falls into one of two cases (Figure 2, bottom):

- if $W=Z$, it is (X,Y,Z,Z,Z): three consecutive Z's at positions $2,3,4$, with $\{X,Y\}=\{1,2,3\}\setminus\{Z\}$;

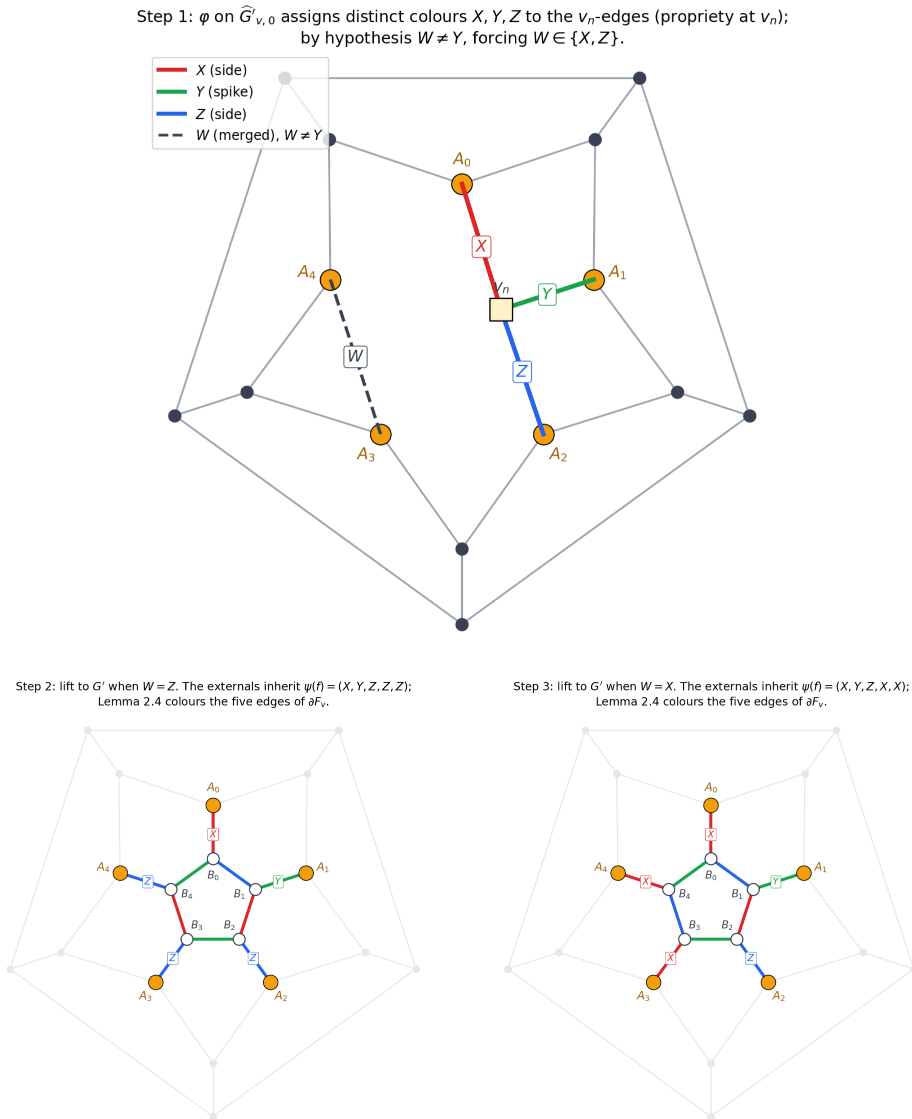

FIGURE 2. The proof of Lemma 3.7, illustrated for $i = 0$ on $G' =$ the dodecahedron. Top: under the assumption $W \neq Y$, propriety at v_n forces $W \in \{X, Z\}$. Bottom: in either case the lift to G' has externals satisfying the hypothesis of Lemma 3.5, which colours ∂F_v to extend ψ to a proper 3-edge-colouring of G'.

- if $W = X$, it is (X, Y, Z, X, X): three consecutive X's at positions $3, 4, 0$, with $\{Y, Z\} = \{1, 2, 3\} \setminus \{X\}$.

Each case satisfies the hypothesis of Lemma 3.5; its $(\Leftarrow)$ direction therefore assigns colours to the boundary edges $B_k B_{k+1}$ that make ψ proper at every B_k.

The resulting ψ is a proper 3-edge-colouring of G': proper at every B_k by the lemma, at every A_k by the construction, and at every other vertex because such a vertex has the same neighbourhood in G' as in $\widehat{G}'_{v,0}$ with the same incident-edge colours. By Tait's theorem, G' is 3-edge-colourable iff G is 4-vertex-colourable, contradicting that G is a counterexample. The assumption $W \neq Y$ is therefore false. $\square$

For a pair of colours $\{a, b\} \subseteq \{1, 2, 3\}$, the subgraph of $\widehat{G}'_{v,i}$ on the edges coloured a or b is 2-regular (since at each vertex exactly one of the three incident edges is excluded), and hence a disjoint union of cycles. We call each such cycle a $\{a, b\}$-*Kempe cycle*, and reserve the notation for the specific cycle containing a given edge when the context makes it clear. Swapping the two colours on a single Kempe cycle yields another proper 3-edge-colouring of the same graph.

Lemma 3.8 (Kempe cycles through the spike)**.** *Let G be a minimal counterexample, fix a reduced dual $\widehat{G}'_{v,i}$ of G', and let φ be a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Write c for the common colour assigned by φ to the spike and the merged edge (Lemma 3.7), and c_0, c_1 for the colours of the side-0 and side-1 edges respectively, so $\{c, c_0, c_1\} = \{1, 2, 3\}$. Then*

1. *the $\{c, c_0\}$-Kempe cycle through the spike edge contains both the side-0 edge and the merged edge;*
2. *the $\{c, c_1\}$-Kempe cycle through the spike edge contains both the side-1 edge and the merged edge.*

Proof. We prove (1); (2) is the same argument with c_1 and the side-1 edge in place of c_0 and the side-0 edge.

The spike edge $\{A_{i+1}, v_n\}$ and the side-0 edge $\{A_i, v_n\}$ share the vertex v_n and receive the two colours c, c_0, so they both lie on the $\{c, c_0\}$-Kempe cycle through v_n. Suppose for contradiction that the merged edge lies on a different $\{c, c_0\}$-Kempe cycle K (it lies on *some* such cycle, since it has colour c). Let φ' be obtained from φ by swapping the colours c and c_0 along K alone: this is a Kempe swap, so φ' is again a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Under φ' the spike edge — which is not on K — still has colour c, but the merged edge — which is on K — now has colour c_0. Hence in φ' the spike and the merged edge receive distinct colours, contradicting Lemma 3.7 applied to φ'. $\square$

4. Edge suppression

We recall the classical operation of *edge suppression* on cubic plane graphs: delete the edge and smooth the two resulting degree-2 endpoints. Under planar duality this coincides with simple-graph contraction on the dual side. It will be the central tool in Section 5 below, where we formulate a sufficient condition for the Four Colour Theorem.

Definition 4.1 (Edge suppression)**.** Let H be a cubic plane graph and $e = uv$ an edge of H with $u \neq v$ and no edge of H parallel to e. The *edge suppression* of H at e is the graph H' obtained in two steps:

1. *Delete* the edge e; the endpoints u and v each drop to degree 2.
2. *Smooth* each of u and v: at u, replace u and its two remaining incident edges ua, ub by a single new edge ab; do the same at v. Both vertices u and v are removed, and two new edges are added in their place.

Provided the smoothings do not introduce a loop or parallel edge, H' is again a cubic plane graph, with $|V(H')| = |V(H)| - 2$ and $|E(H')| = |E(H)| - 3$.

Equivalently, H' is the planar dual of $\mathrm{dual}(H)/e^*$, where e^* is the edge of $\mathrm{dual}(H)$ crossing e and the contraction on the right-hand side is simple-graph contraction (loops removed, parallel edges absorbed). Under planar duality, contracting e^* in $\mathrm{dual}(H)$ merges the two triangular faces of $\mathrm{dual}(H)$ incident to e^*, and the parallel-edge cleanup corresponds exactly to the smoothing step on the primal side.

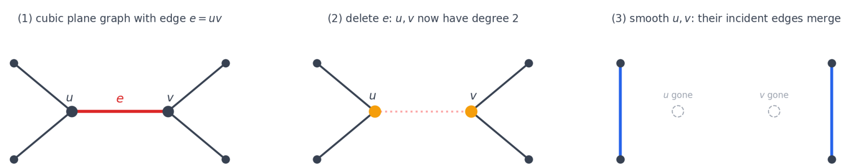

FIGURE 3. Edge suppression (Definition 4.1). Left: a fragment of a cubic plane graph with the suppressed edge $e = uv$ highlighted in red. Middle: deleting e leaves u and v of degree 2. Right: smoothing u and v replaces each pair of incident edges by a single new edge, removing u, v and giving a cubic plane graph again.

Theorem 4.2 (Edge suppression across a 4-face preserves 3-edge-colourability)**.** *Let H be a cubic plane graph with a proper 3-edge-colouring φ, let f be a face of H with $|\partial f| = 4$, and let e_0, e_1 be the two edges of ∂f sharing no endpoint (the opposite pair on the 4-cycle ∂f). If $\varphi(e_0) \neq \varphi(e_1)$ and the edge suppression of H at e_0 (Definition 4.1) is well-defined (no loops or parallel edges are created), then the suppressed graph admits a proper 3-edge-colouring.*

Proof. Write ∂f as the 4-cycle $v_0v_1v_2v_3$ with $e_0 = v_0v_1$ and $e_1 = v_2v_3$ (so e_0, e_1 are opposite); the remaining two boundary edges of f are $e_2 := v_1v_2$ and $e_3 := v_3v_0$. Since H is cubic, each v_i has exactly one edge not on ∂f: write w_i for that edge and u_i for its other endpoint, so $w_i = v_iu_i$ with $u_i \notin \{v_0, v_1, v_2, v_3\}$, for each $i \in \{0, 1, 2, 3\}$. Put $a := \varphi(e_0)$, $b := \varphi(e_1)$, and let c be the third colour.

Forced colours on the face. Propriety at v_1 and v_2 forces $\varphi(e_2) \notin \{a, b\}$, so $\varphi(e_2) = c$; then $\varphi(w_1) = b$ and $\varphi(w_2) = a$. Symmetrically $\varphi(e_3) = c$, $\varphi(w_0) = b$, and $\varphi(w_3) = a$. In particular $\varphi(w_0) = \varphi(w_1) = b$.

Construction of φ'. Let H' denote the edge suppression of H at e_0; its new edges are $e_3' := v_3u_0$ (replacing e_3 and w_0 via the smoothing at v_0) and $e_2' := v_2u_1$ (replacing e_2 and w_1 via the smoothing at v_1). Define $\varphi' \colon E(H') \to \{1, 2, 3\}$ by

$$\varphi'(e) := \begin{cases} c & \text{if } e = e_1, \\ b & \text{if } e \in \{e_2', e_3'\}, \\ \varphi(e) & \text{otherwise.} \end{cases}$$

That is: give each smoothed-in edge the colour b (the colour of the two w_i it absorbs), recolour e_1 to c, and leave every other edge of H' with its φ-colour.

Propriety. Every vertex of H' other than v_2, v_3, u_0, u_1 has the same incident edges and the same φ'-colours as it did under φ, so propriety is inherited there. At the four affected vertices,

vertex	edges in H'	colours under φ'
v_2	$e_1,\ w_2,\ e_2'$	$c,\ a,\ b$
v_3	$e_1,\ w_3,\ e_3'$	$c,\ a,\ b$
u_0	$e_3',\ \alpha_0,\ \beta_0$	$b,\ a,\ c$
u_1	$e_2',\ \alpha_1,\ \beta_1$	$b,\ a,\ c$

where α_i, β_i are the two edges of H at u_i other than w_i, whose φ-colours are forced to $\{a, c\}$ by propriety at u_i (since $\varphi(w_i) = b$). Each row lists three distinct colours, so φ' is proper. □

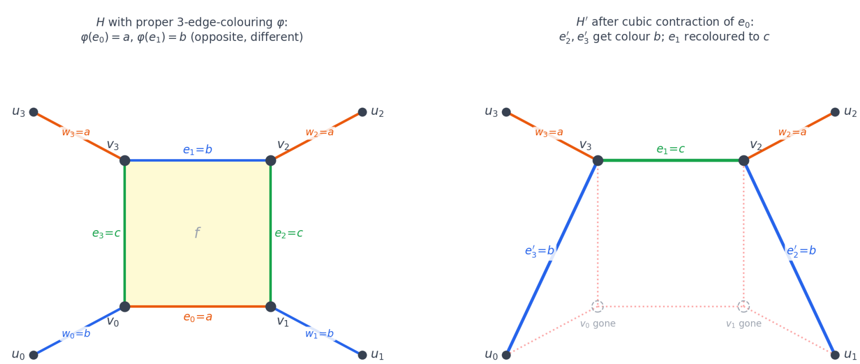

FIGURE 4. The recolouring used in the proof of Theorem 4.2. Left: the 4-face f of H under φ, with the forced colours $\varphi(e_0) = a$, $\varphi(e_1) = b$, $\varphi(e_2) = \varphi(e_3) = c$, $\varphi(w_0) = \varphi(w_1) = b$, and $\varphi(w_2) = \varphi(w_3) = a$. Right: the suppressed graph H' under φ'. The smoothed-in edges e_2', e_3' inherit the colour b from w_0, w_1, and e_1 is recoloured from b to c; every edge outside the face neighbourhood keeps its φ-colour (dotted in red: the five edges of H removed by the suppression).

5. The face-monochromatic-pair conjecture and the Four Colour Theorem

The following conjecture identifies a structural property of every proper 3-edge-colouring of a reduced dual of a minimal counterexample. If true, it implies the Four Colour Theorem via Theorem 4.2.

Conjecture 5.1 (Face-monochromatic-pair conjecture)**.** *Let G be a minimal counterexample to the Four Colour Theorem, and let $\widehat{G}'_{v,i}$ be a reduced dual of $G' =$ dual(G). Then for every proper 3-edge-colouring φ of $\widehat{G}'_{v,i}$ there exist a face F of $\widehat{G}'_{v,i}$ and two distinct edges $e_1, e_2 \in \partial F$, with neither e_1 nor e_2 equal to the merged edge, such that:*

(1) $\varphi(e_1) = \varphi(e_2)$. *Write $a := \varphi(e_1) = \varphi(e_2)$.*

(2) *e_1, e_2, and the merged edge all lie on a common $\{a,b\}$-Kempe cycle of φ, for some colour $b \neq a$.*

(3) *Exactly one edge of ∂F lies between e_1 and e_2 along one of the two arcs of ∂F. Equivalently, subdividing e_1, e_2 by new vertices X_1, X_2 and joining them by a new edge X_1X_2 inside F creates a new face f_n bounded by exactly 4 edges (the new edge X_1X_2, the two subdivision halves adjacent to it, and the single ∂F-edge between e_1 and e_2).*

Lemma 5.2 (A Heawood-constant Kempe cycle does not admit the clause-(3) arc)**.** *Let G be a minimal counterexample to the Four Colour Theorem, fix a reduced dual $\widehat{G}'_{v,i}$ of $G' = \mathrm{dual}(G)$, and let φ be a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Set $a := \varphi(\mathrm{merged})$ and let K be the $\{a,b\}$-Kempe cycle of φ through the merged edge for some $b \in \{1,2,3\} \setminus \{a\}$. If h_φ is constant on $V(K)$, then no edge $e \in E(K)$ admits a face F of $\widehat{G}'_{v,i}$ and two non-incident edges $e_1, e_2 \in \partial F$ such that $\varphi(e_1) = \varphi(e_2)$ and e is the unique edge of ∂F between e_1 and e_2 along one of the two arcs of ∂F — that is, no edge of K admits the clause-(3) arc of Conjecture 5.1 with e_1, e_2 at its two endpoints.*

Proof. Let c be the third colour. Fix any edge $e \in E(K)$ joining $v_0, v_1 \in V(K)$. By hypothesis $h_\varphi(v_0) = h_\varphi(v_1)$; after possibly relabelling we may take $h_\varphi(v_0) = h_\varphi(v_1) = +1$, so by Definition 3.1 the clockwise cyclic colour order at v_0 and at v_1 is the same even cyclic class (a,b,c).

Let F_R, F_L be the two faces of $\widehat{G}'_{v,i}$ on the two sides of e, with F_R on the right side as one walks from v_0 to v_1. For a vertex $v \in \{v_0, v_1\}$, the non-e edge of ∂F_R at v is the next-clockwise edge from e around v_0 (since at v_0 the right side coincides with the clockwise next edge from e) and the next-counter-clockwise edge from e around v_1 (since at v_1 the orientation of e is reversed, so the right side coincides with the counter-clockwise next edge from e).

Case A: $\varphi(e) = a$. In the CW order (a,b,c) at v_0 the next-CW edge from e has colour b; in the same CW order (a,b,c) at v_1 the next-CCW edge from e has colour c (since CCW-next from a in cyclic order (a,b,c) is c). Hence the non-e edge of ∂F_R at v_0 has colour b, while the non-e edge of ∂F_R at v_1 has colour c — these differ. Symmetrically, the non-e edges of ∂F_L at v_0 and v_1 have colours c and b respectively, again different. Hence the colour-b edges at v_0 and v_1 lie on opposite faces of e, and the same for the colour-c edges; no face of $\widehat{G}'_{v,i}$ contains two same-coloured non-e edges at $\{v_0, v_1\}$.

Case B: $\varphi(e) = b$. By the analogous reasoning, the non-e edges of ∂F_R at v_0 and v_1 have colours c and a respectively, and those of ∂F_L have colours a and c. The colour-a edges at v_0, v_1 thus lie on opposite faces of e, and so do the colour-c edges.

In either case, no face F of $\widehat{G}'_{v,i}$ has two same-coloured non-e edges at $\{v_0, v_1\}$ on ∂F, so the clause-(3) arc (with e as the unique ∂F-edge between e_1 and e_2 at the endpoints of e) cannot be realised. □

Lemma 5.3 (If Conjecture 5.1 fails, both Kempe cycles through merged have constant Heawood number)**.** *Let G, $\widehat{G}'_{v,i}$, φ be as in Lemma 5.2, set $a := \varphi(\mathrm{merged})$, and let K_b, K_c be the two Kempe cycles of φ through the merged edge — the $\{a,b\}$-Kempe cycle and the $\{a,c\}$-Kempe cycle, where $\{b,c\} = \{1,2,3\} \setminus \{a\}$. If no triple*

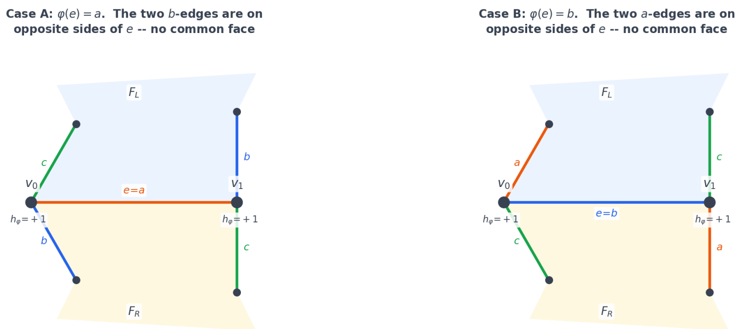

FIGURE 5. The two cases in the proof of Lemma 5.2. Vertices v_0, v_1 are consecutive on the $\{a, b\}$-Kempe cycle K, joined by an edge e, with the lemma's hypothesis $h_\varphi(v_0) = h_\varphi(v_1) = +1$ — so both vertices share the clockwise colour order (a, b, c). *Left (Case A):* when $\varphi(e) = a$, the colour-b edge at v_0 lies south of e (on ∂F_R) and the colour-b edge at v_1 lies north of e (on ∂F_L); the two would-be witness edges are on opposite faces, so no face of $\widehat{G}'_{v,i}$ contains both. *Right (Case B):* when $\varphi(e) = b$, the colour-a edges at v_0, v_1 are likewise on opposite sides of e. In either case the clause-(3) arc of Conjecture 5.1 cannot be realised at e.

(F, e_1, e_2) satisfies clauses (1)–(3) of Conjecture 5.1 on $(G, \widehat{G}'_{v,i}, \varphi)$, then h_φ is constant on $V(K_b)$ and on $V(K_c)$, and the two constants agree (so all of $V(K_b) \cup V(K_c)$ shares a common Heawood number).

Proof. We prove the contrapositive: if h_φ is non-constant on $V(K_b)$ (the argument for K_c is identical), then a triple (F, e_1, e_2) realising clauses (1)–(3) of Conjecture 5.1 exists. The argument is precisely the case analysis of Lemma 5.2 run with the opposite Heawood hypothesis.

Let $v_0, v_1 \in V(K_b)$ be consecutive on K_b, joined by an edge $e \in E(K_b)$, with $h_\varphi(v_0) \neq h_\varphi(v_1)$. After possibly swapping take $h_\varphi(v_0) = +1$ and $h_\varphi(v_1) = -1$, so by Definition 3.1 the clockwise cyclic colour order at v_0 is the even class (a, b, c) and at v_1 is the odd class (a, c, b).

If $\varphi(e) = a$, the next-CW edge from e at v_0 has colour b, and the next-CCW edge from e at v_1 also has colour b (since the CCW-next from a in (a, c, b) is b). Both these b-edges lie on ∂F_R, where F_R is the face on the right of e walking $v_0 \to v_1$; e is the unique ∂F_R-edge between them on one arc. Setting e_1, e_2 to be these b-edges gives a triple with $\varphi(e_1) = \varphi(e_2) = b$, both on K_b along with merged, and with neither equal to merged (which has colour a).

If $\varphi(e) = b$, the symmetric argument places the colour-a edges at v_0, v_1 on ∂F_L with e between them; choosing (v_0, v_1) so that neither is an endpoint of merged (possible since at most two K_b-vertices — the endpoints of merged — could force this issue, and a non-constant h_φ on K_b guarantees a differing-Heawood pair away from them) yields the witness.

Either way (F, e_1, e_2) contradicts the hypothesis, so h_φ must be constant on $V(K_b)$. The same argument with K_c in place of K_b gives constancy on $V(K_c)$.

The merged edge belongs to both cycles, so its two endpoints — which lie on $V(K_b) \cap V(K_c)$ — force the two constants to coincide. □

Corollary 5.4 (Per-cycle form)**.** *Let G, $\widehat{G}'_{v,i}$, φ be as in Lemma 5.3, and let K be either of the two Kempe cycles of φ through the merged edge. If h_φ is not constant on $V(K)$, then a triple (F, e_1, e_2) satisfying clauses (1)–(3) of Conjecture 5.1 on $(G, \widehat{G}'_{v,i}, \varphi)$ exists.*

Proof. This is precisely the case analysis used to prove Lemma 5.3: applied to any consecutive pair of vertices on K with differing Heawood numbers, the construction in that proof produces a clauses-(1)–(3) witness without ever needing to inspect the other Kempe cycle. □

Conjecture 5.5 (Constant Heawood on two edge-sharing Kempe cycles, large-face cubic plane graphs — **FALSE**)**.** *Let H be a cubic plane graph in which every face has length at least* 5*, with a proper* 3*-edge-colouring φ. Fix a colour $a \in \{1,2,3\}$ and let $\{b,c\} = \{1,2,3\} \setminus \{a\}$. Let K_0 be an $\{a,b\}$-Kempe cycle of φ and K_1 an $\{a,c\}$-Kempe cycle of φ such that $E(K_0) \cap E(K_1) \neq \emptyset$ (equivalently, K_0 and K_1 share at least one colour-a edge). If h_φ is constant on $V(K_0)$, then h_φ is* not *constant on $V(K_1)$.*

Remark 5.6 (Disproof of Conjecture 5.5)*.* Conjecture 5.5 is *false.* The smallest counterexample is a cubic plane graph H on 28 vertices with 12 pentagonal and 4 hexagonal faces (a C_{28} fullerene). It is the planar dual of the third element (in `Sage`'s order) of `graphs.triangulations(16, minimum_degree=5)`, with canonical `graph6` string

`[kG[A?_A?_?_?K?D?@_CO?o?@_??A??@C??O??AG?C???¿???a???W???A_???F.`

A proper 3-edge-colouring of H (colours red/blue/green, see Figure 6) makes both

$$
\begin{aligned}
K_{\text{red,blue}} &= \text{a 12-cycle on } H, \\
K_{\text{red,green}} &= \text{a different 12-cycle on } H,
\end{aligned}
$$

sharing the colour-red edge $(0,1)$ and satisfying $h_\varphi \equiv -1$ on the vertex set of each. Globally h_φ takes value $+1$ on 4 vertices and -1 on 24; the four $+1$-vertices and a further four lie outside $V(K_0) \cup V(K_1)$, which has size 20. The construction, verification, and rendering are in `experiments/verify_28_vertex_counterexample.py`, and the exhaustive search that found it is in `experiments/search_min_face5_counterexample.py`.

The face-length-≥ 5 hypothesis is in fact the strongest face-length hypothesis admitting any cubic plane graphs at all on the sphere: by Euler's formula, if every face had length ≥ 6 then the sum of face lengths would satisfy $3|V(H)| \geq 6(|V(H)|/2 + 2)$, i.e. $0 \geq 12$, contradiction. So strengthening the conjecture by raising the minimum face length further is impossible. Without the face-length hypothesis there are far smaller counterexamples, including the tetrahedron K_4 at $|V| = 4$ (every Kempe cycle is a 4-cycle visiting every vertex, with h_φ constant on all of them by vertex-transitivity) and an 8-vertex example (graph6 `G}GOW[`) found by `experiments/search_smaller_counterexample.py`; both have girth 3. An ad-hoc 40-vertex counterexample with the same "two intersecting Kempe cycles $\equiv -1$, large region outside" flavour is in `constant_heawood_counterexample.tikz`.

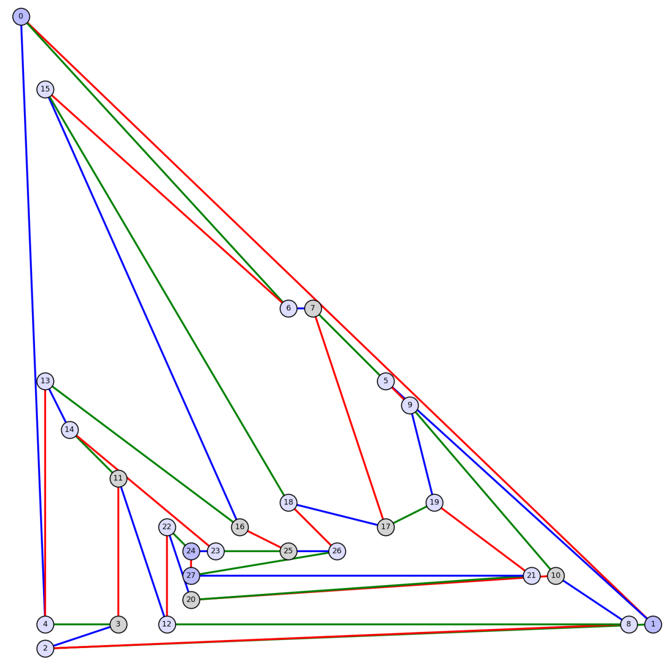

FIGURE 6. Smallest counterexample to Conjecture 5.5: a C_{28} fullerene-style cubic plane graph (12 pentagons + 4 hexagons) with a proper 3-edge-colouring on which h_φ is simultaneously constant ($\equiv -1$) on the red/blue 12-cycle and the red/green 12-cycle, which share the colour-red edge $(0, 1)$. Light-shaded nodes are on $V(K_0) \cap V(K_1)$; medium-shaded on $V(K_0) \cup V(K_1) \setminus V(K_0) \cap V(K_1)$; grey on neither.

A reduction of Conjecture 5.1 via Heawood's face-sum identity. The empirical work of Section 3 (the 0/142,812 result on chord-apex+Kempe colourings, recorded in Remark 5.20) suggests a structural proof strategy via the classical Heawood face-sum identity [1]:

$$\sum_{v \in \partial f} h_\varphi(v) \;\equiv\; 0 \pmod 3 \qquad \text{for every face } f \text{ of } H, \tag{5.1}$$

which holds for any proper 3-edge-colouring φ of any cubic plane graph H.

Conjecture 5.7 (Deciding face)**.** *Let G be a minimal counterexample to the Four Colour Theorem, let $\widehat{G}'_{v,i}$ be a reduced dual of G', and let φ be a chord-apex+Kempe colouring of $\widehat{G}'_{v,i}$. Write K_b and K_c for the two Kempe cycles through the spike edge (Lemma 3.8). Then $\widehat{G}'_{v,i}$ has a face f satisfying*

$$\partial f \subseteq V(K_b) \cup V(K_c) \qquad \textit{and} \qquad |f| \not\equiv 0 \pmod 3.$$

Theorem 5.8. *Conjecture 5.7 implies Conjecture 5.1.*

Proof. We argue by contradiction. Suppose Conjecture 5.1 fails: there exist a minimal counterexample G, a reduced dual $\widehat{G}'_{v,i}$, and a chord-apex+Kempe colouring φ of $\widehat{G}'_{v,i}$ admitting no clauses-(1)-(3) witness of Conjecture 5.1.

By Lemma 5.3, the absence of any clauses-(1)-(3) witness forces h_φ to be constant on $V(K_b) \cup V(K_c)$; write the common value as $\varepsilon \in \{+1, -1\}$.

By Conjecture 5.7, there is a face f of $\widehat{G}'_{v,i}$ with $\partial f \subseteq V(K_b) \cup V(K_c)$ and $|f| \not\equiv 0 \pmod 3$. Applying (5.1) to f and using $h_\varphi(v) = \varepsilon$ for every $v \in \partial f$:

$$\sum_{v \in \partial f} h_\varphi(v) \;=\; \varepsilon \cdot |f| \;\equiv\; 0 \pmod 3.$$

Since 3 is prime and $\gcd(|f|, 3) = 1$, we obtain $\varepsilon \equiv 0 \pmod 3$. But $\varepsilon \in \{+1, -1\}$, so $\varepsilon \not\equiv 0 \pmod 3$ — contradiction. □

A partial structural proof of Conjecture 5.7. We single out a specific candidate face — the "flank" face that the spike, side-0 edge, and one boundary arc of the new pentagonal hole together bound — and prove the deciding-face property for it when the adjacent G'-face is pentagonal or hexagonal. This is the case for every reduced dual where at least one of v's neighbours B_i or B_{i+1} in the parent triangulation G has degree 5 or 6 — e.g., for the icosahedron and the dodecahedron's other near neighbours.

Definition 5.9 (Flank face)**.** Fix a reduced dual $\widehat{G}'_{v,i}$ and let F be the post-deletion pentagonal hole. Order the five outer endpoints A_0, A_1, A_2, A_3, A_4 in clockwise order along ∂F as in Definition 3.2. The *lower flank face* $F^\flat_{i,i+1}$ of $\widehat{G}'_{v,i}$ is the face of $\widehat{G}'_{v,i}$ whose boundary is the closed walk

$$v_n \xrightarrow{\text{side-0}} A_i \xrightarrow{\partial F\text{-arc}} A_{i+1} \xrightarrow{\text{spike}} v_n,$$

i.e. the side-0 edge, followed by the maximal sub-arc of ∂F from A_i to A_{i+1} not crossing v_n or the merged edge, followed by the spike edge in reverse. The *upper flank face* $F^\flat_{i+1,i+2}$ is defined analogously with side-1 replacing side-0 and $A_{i+1} \to A_{i+2}$ replacing $A_i \to A_{i+1}$.

Lemma 5.10 (Flank-length formula)**.** $|F^\flat_{i,i+1}| = n_i - 1$, *where n_i is the length of the G'-face F_i adjacent to F_v across the edge (B_i, B_{i+1}) in the pre-reduction dual G'. (Symmetrically, $|F^\flat_{i+1,i+2}| = n_{i+1} - 1$.)*

Proof. After step (1) of Definition 3.2 the dual face F_i, originally a closed walk of length n_i around the boundary edges (B_i, B_{i+1}), (B_{i+1}, A_{i+1}), ∂F_i interior, and (A_i, B_i), becomes (after deletion of B_i, B_{i+1} and their three incident edges) an open arc from A_{i+1} to A_i of length $n_i - 3$ in the post-deletion graph. The lower flank face $F^\flat_{i,i+1}$ adds back the side-0 edge (v_n, A_i) (length 1) and the spike edge (v_n, A_{i+1}) (length 1). Its total boundary length is therefore $(n_i - 3) + 1 + 1 = n_i - 1$. □

Lemma 5.11 (Flank covering, base case $n_i = 5$)**.** *If $n_i = 5$ then $\partial F^\flat_{i,i+1} \subseteq V(K_b) \cup V(K_c)$.*

Proof. When $n_i = 5$ the boundary of $F^\flat_{i,i+1}$ visits exactly four vertices: v_n, A_i, a single intermediate P, and A_{i+1}. We have $v_n, A_i, A_{i+1} \in V(K_b) \cup V(K_c)$ by Lemma 3.8, so it remains to show $P \in V(K_b) \cup V(K_c)$.

The vertex A_{i+1} has degree 3, with one of its three incident edges being the spike (colour c) and the other two having colours $\{c_0, c_1\}$ (one each, by properness). Lemma 3.8 gives $A_{i+1} \in V(K_b) \cap V(K_c)$, so K_b uses the spike together with A_{i+1}'s colour-c_0 edge, and K_c uses the spike together with A_{i+1}'s colour-c_1 edge. The two non-spike edges at A_{i+1} are exactly $A_{i+1}P$ (the boundary edge into the lower flank) and $A_{i+1}P'$ (the boundary edge into the upper flank, $P' \in F^{\flat}_{i+1,i+2}$). Whichever of these is $A_{i+1}P$, its colour is in $\{c_0, c_1\}$, so the corresponding Kempe cycle (K_b if colour c_0, K_c if colour c_1) walks $A_{i+1} \to P$, placing P in that cycle's vertex set. $\square$

Lemma 5.12 (Flank covering, $n_i = 6$). *If $n_i = 6$ then $\partial F^{\flat}_{i,i+1} \subseteq V(K_b) \cup V(K_c)$.*

Proof. The boundary now visits five vertices: v_n, A_i, P_1, P_2, A_{i+1}, with P_1 adjacent to A_i and P_2 adjacent to A_{i+1}. The named vertices and P_2 are in $V(K_b) \cup V(K_c)$ by Lemma 5.11's argument applied at A_{i+1}. It remains to place P_1.

Case (a): $\varphi(A_iP_1) = c$. Then K_b's walk uses A_i's colour-c edge (along with side-0), so K_b walks $A_i \to P_1$ and $P_1 \in V(K_b)$.

Case (b): $\varphi(A_iP_1) = c_1$. Then A_i's colour-c edge is the *other* non-side-0 edge of A_i, namely A_iQ where Q is the intermediate on $\partial F^{\flat}_{i-1,i}$ adjacent to A_i. Now consider P_2. By the analysis above we have $\varphi(A_{i+1}P_2) \in \{c_0, c_1\}$, and properness at P_2 gives $\varphi(P_2)$'s three incident edges colours $\{c_0, c, c_1\}$ in some order, with $\varphi(A_{i+1}P_2)$ accounting for one of $\{c_0, c_1\}$. The remaining two colours, c and (c_1 or c_0), are split between $\varphi(P_1P_2)$ and $\varphi(P_2O_2)$ where O_2 is P_2's third (outer) neighbour. Properness at P_1 (which has $\varphi(A_iP_1) = c_1$ already by Case (b) hypothesis) forbids $\varphi(P_1P_2) = c_1$. Hence

$$\varphi(P_1P_2) \; \in \; \{c, c_0\}.$$

Now $P_2 \in V(K_b) \cup V(K_c)$ as shown, and at P_2 the cycle that contains P_2 uses two specific edges of P_2. If $P_2 \in V(K_b)$ the cycle uses P_2's colour-c and colour-c_0 edges; if $P_2 \in V(K_c)$ it uses colour-c and colour-c_1. In either case the cycle at P_2 uses P_2's colour-c edge. By the preceding paragraph $\varphi(P_1P_2) \in \{c, c_0\}$, so the $\{c, \varphi(P_2P_1)\}$-Kempe cycle through P_2 passes from P_2 to P_1. If this cycle is K_b, $P_1 \in V(K_b)$; if it is K_c (which requires $\varphi(P_1P_2) = c$, since K_c uses $\{c, c_1\}$), then $P_1 \in V(K_c)$.

In either case $P_1 \in V(K_b) \cup V(K_c)$. $\square$

Theorem 5.13 (Partial proof of Conjecture 5.7 via flank face). *If $n_i \in \{5, 6\}$ or $n_{i+1} \in \{5, 6\}$ for the reduction index i, then $\widehat{G}'_{v,i}$ has a deciding face: one of the two flank faces $F^{\flat}_{i,i+1}$ or $F^{\flat}_{i+1,i+2}$.*

Proof. Without loss of generality $n_i \in \{5, 6\}$ (the n_{i+1} case is symmetric, swapping side-0 with side-1). By Lemma 5.10, $|F^{\flat}_{i,i+1}|$ is 4 (if $n_i = 5$) or 5 (if $n_i = 6$); both are $\not\equiv 0 \pmod 3$. By Lemmas 5.11 and 5.12, $\partial F^{\flat}_{i,i+1} \subseteq V(K_b) \cup V(K_c)$. Hence $F^{\flat}_{i,i+1}$ is a deciding face. $\square$

We extend the partial proof to handle configurations where neither flank face qualifies (i.e., both $n_i, n_{i+1} \geq 7$). The candidate is the *outer face* inside F, on the merged side of v_n.

Definition 5.14 (Outer face). The *outer face* $F^{\flat}_{\text{outer}}$ of $\widehat{G}'_{v,i}$ is the face whose boundary is the closed walk

$$v_n \xrightarrow{\text{side-1}} A_{i+2} \xrightarrow{\partial F\text{-arc}} A_{i+3} \xrightarrow{\text{merged}} A_{i+4} \xrightarrow{\partial F\text{-arc}} A_i \xrightarrow{\text{side-0}} v_n,$$

i.e., the face inside F bounded by side-1, the $A_{i+2} \to A_{i+3}$ arc of ∂F, the merged edge, the $A_{i+4} \to A_i$ arc of ∂F, and side-0 reversed.

Lemma 5.15 (Outer-face length formula). $|F^{\flat}_{\text{outer}}| = n_{i+2} + n_{i+4} - 3.$

Proof. The boundary length is 1 (side-1) $+(n_{i+2}-3)$ (arc $A_{i+2} \to A_{i+3}$) $+1$ (merged) $+(n_{i+4}-3)$ (arc $A_{i+4} \to A_i$) $+1$ (side-0) $= n_{i+2}+n_{i+4}-3$. ☐

Lemma 5.16 (Outer-face covering, pentagonal-flanks case)**.** *If* $n_{i+2} = n_{i+4} = 5$ *then* $\partial F^\flat_{\mathrm{outer}} \subseteq V(K_b) \cup V(K_c)$.

Proof. The boundary visits v_n, A_{i+2}, a single intermediate P_{23} on the $A_{i+2} \to A_{i+3}$ arc (since $n_{i+2} = 5$ gives arc length 2), A_{i+3}, A_{i+4}, a single intermediate P_{40} on the $A_{i+4} \to A_i$ arc, and A_i. The five named vertices are in $V(K_b) \cup V(K_c)$ by Lemma 3.8; we need only place P_{23} and P_{40}.

By Lemma 3.8, $A_{i+3}, A_{i+4} \in V(K_b) \cap V(K_c)$ via the merged edge. Apply the argument of Lemma 5.11 at A_{i+3}: A_{i+3} has three incident edges, one being the merged edge (colour c) and the other two with colours $\{c_0, c_1\}$. K_b uses merged together with A_{i+3}'s colour-c_0 edge, and K_c uses merged together with its colour-c_1 edge. The two non-merged edges at A_{i+3} are $A_{i+3}P_{23}$ and $A_{i+3}P'$ (where P' lies on the merged-side face $F^\flat_{\mathrm{merged}}$). Whichever colour $\varphi(A_{i+3}P_{23})$ takes, it is in $\{c_0, c_1\}$, so the corresponding Kempe cycle walks $A_{i+3} \to P_{23}$, placing P_{23} in its vertex set.

Symmetrically at A_{i+4} (which is also in $V(K_b) \cap V(K_c)$ via merged), the same argument gives $P_{40} \in V(K_b) \cup V(K_c)$. ☐

Theorem 5.17 (Extended partial proof of Conjecture 5.7)**.** *The deciding face exists for* $\widehat{G}'_{v,i}$ *whenever at least one of the following holds:* ***(a)*** $n_i \in \{5, 6\}$*;* ***(b)*** $n_{i+1} \in \{5, 6\}$*; or* ***(c)*** $n_{i+2} = n_{i+4} = 5$*.*

Proof. Cases (a) and (b) are covered by Theorem 5.13, yielding the flank face $F^\flat_{i,i+1}$ or $F^\flat_{i+1,i+2}$ as the deciding face. For case (c), $|F^\flat_{\mathrm{outer}}| = 5+5-3 = 7 \equiv 1 \pmod 3$ by Lemma 5.15, and $\partial F^\flat_{\mathrm{outer}} \subseteq V(K_b) \cup V(K_c)$ by Lemma 5.16. Hence $F^\flat_{\mathrm{outer}}$ is a deciding face. ☐

Remark 5.18 (Empirical coverage of Theorem 5.17). Across all 1,586 (G, v) pairs underlying chord-apex+Kempe colourings for $|V(G)| \le 20$, every cyclic neighbour-degree sequence around v contains at least one degree-5 entry (see `experiments/check_v_neighbour_degrees.py`). Of the 7,930 (G, v, i) triples we have:

- 7,906 (99.70%) satisfy case (a) or (b);
- the remaining 24 (0.30%) have $n_i, n_{i+1} \ge 7$, but *all* 24 have $(n_{i+2}, n_{i+3}, n_{i+4}) = (5, 5, 5)$, so they satisfy case (c).

Combining Theorems 5.8 and 5.17, this yields a **structural proof of Conjecture 5.1 on every chord-apex+Kempe colouring of every reduced dual with** $|V(G)| \le 20$**.**

The structurally open remaining case is configurations where *both* $n_i, n_{i+1} \ge 7$ *and* the merged-side degrees (n_{i+2}, n_{i+4}) are not both 5. We have not observed this case empirically for $|V(G)| \le 20$, but it could arise for larger triangulations and would then need a further covering lemma (extending Lemma 5.16 to $(n_{i+2}, n_{i+4}) \in \{5, 6\}^2$, or a separate argument via $F^\flat_{\mathrm{merged}}$ of length $n_{i+3} - 2$).

Remark 5.19 (Empirical verification of Conjecture 5.7). We have verified Conjecture 5.7 computationally on every chord-apex+Kempe colouring of every reduced dual of every triangulation G of minimum degree 5 with $|V(G)| \le 20$. Across 142,812 such colourings, every single one admits a deciding face. The empirical face-length distribution is:

$\lvert f\rvert$	$\lvert f\rvert \bmod 3$	#colourings
4	1	13,074
5	2	102,498
7	1	18,570
8	2	7,752
10	1	846
11	2	72

(Every length appearing is $\not\equiv 0 \pmod 3$, as required.) See `experiments/check_deciding_face.py`.

A natural candidate for the deciding face is one of the two "flank" faces of the reduced dual — the face bounded by the side-0 edge, the $A_i \to A_{i+1}$ arc of ∂F, and the spike edge, or its $A_{i+1} \to A_{i+2}$ counterpart. When the parent triangulation's vertex v has a degree-5 neighbour B_k (i.e. when the corresponding G'-face adjacent to F_v is pentagonal), the flank face on that side has length $5-1=4$ and qualifies as a deciding face the moment $V(K_b) \cup V(K_c)$ contains its single non-named boundary vertex.

Remark 5.20 (Empirical near-proof of Conjecture 5.1 via Corollary 5.4)*.* By Corollary 5.4, Conjecture 5.1 follows from the (a priori weaker) structural claim: *for every chord-apex+Kempe colouring φ of every reduced dual $\widehat{G}'_{v,i}$, h_φ is not constant on $V(K_b)$ (equivalently, not constant on $V(K_c)$).* We have verified this claim computationally on all chord-apex+Kempe colourings of reduced duals with $|V(G)| \leq 20$ (including the six Holton–McKay duals at $n = 21$ as a special case); see `experiments/check_heawood_on_kempe.py` and `experiments/check_constancy_obstruction.py`.

n	#col. tested	#non-const. on $V(K_b)$	#non-const. on $V(K_c)$	status
14	216	216	216	all non-constant
16	864	864	864	all non-constant
17	4,650	4,650	4,650	all non-constant
18	8,070	8,070	8,070	all non-constant
19	21,138	21,138	21,138	all non-constant
20	107,874	107,874	107,874	all non-constant
total ($n \leq 20$)	142,812	142,812	142,812	

In particular, h_φ is non-constant on $V(K_b)$ alone in every tested colouring (and likewise on $V(K_c)$); by Corollary 5.4 each such colouring admits a Conjecture-5.1 witness. This gives an empirical near-proof of the conjecture for $|V(G)| \leq 20$ independent of (and consistent with) the direct witness-search check of Remark 5.21. A structural proof of non-constancy on $V(K_b)$ (or on $V(K_c)$) would convert this into a proof of Conjecture 5.1 proper.

Remark 5.21*.* The conjecture cannot be tested on actual minimal counterexamples (none exist by the Four Colour Theorem), but its conclusion is checkable on the structural surrogates: proper 3-edge-colourings of reduced duals that satisfy both the chord-apex condition (Lemma 3.7) and the Kempe-cycle conditions (Lemma 3.8), since a minimal counterexample's reduced dual is forced to admit such colourings. For every min-degree-5 triangulation G with $|V(G)| \leq 21$, every pentagonal face F of G', and every reduction index $i \in \{0, \dots, 4\}$, we enumerated all such colourings and verified the three clauses of Conjecture 5.1 (see `experiments/check_conj_face_kempe_scaled.py`); $n = 22$ ran past a 1800 s budget after 641,700 colourings (all pass) without finishing the full set of 651 triangulations.

n	#tri	#col. tested	#sat.	status
12	1	0	—	vacuous (icosahedron)
13	0	—	—	no min-deg-5 tri
14	1	216	216	all pass
15	1	0	—	vacuous
16	3	864	864	all pass
17	4	4,650	4,650	all pass
18	12	8,070	8,070	all pass
19	23	21,138	21,138	all pass
20	73	107,874	107,874	all pass
21	192	392,370	392,370	all pass
22 (part.)	651	641,700	641,700	timeout
total ($n \le 21$)	311	535,182	535,182	

The vacuous rows ($n = 12, 15$) are those where the relevant reduced duals admit no chord-apex+Kempe colourings, so the conjecture has no content there.

The following strengthening adds a fourth clause that arranges the new 4-face f_n to satisfy the hypotheses of Theorem 4.2.

Conjecture 5.22 (Strengthening of Conjecture 5.1)**.** *Let G, $\widehat{G}'_{v,i}$, φ be as in Conjecture 5.1. Then there exist F, e_1, e_2 satisfying clauses (1)–(3) of that conjecture, together with the following additional clause.*

(4) *Let $\widehat{G}'^{+}$ be the modified graph obtained from $\widehat{G}'_{v,i}$ by the modification of clause (3), and let φ' be the proper 3-edge-colouring of $\widehat{G}'^{+}$ obtained from φ by swapping the two colours along the (subdivided) $\{a, b\}$-Kempe cycle of clause (2) and assigning the new edge X_1X_2 the remaining (third) colour c. (Equivalently: φ' agrees with φ on every edge of $\widehat{G}'_{v,i}$ outside that Kempe cycle, and at X_1, X_2 the two subdivision halves take the colours $\{a, b\}$ in the order forced by propriety.) Then either*
   - (i) *∂f_n uses all three colours under φ', or*
   - (ii) *the $\{b, c\}$-Kempe cycle of φ' through X_1X_2 is incident to exactly one edge of ∂f_n (namely X_1X_2 itself).*

Remark 5.23. The strengthened conjecture was tested on the same chord-apex+Kempe colourings as Remark 5.21; for each colouring we sought any Conjecture-5.1-witness (F, e_1, e_2) whose accompanying f_n satisfies clause (4) (see `experiments/check_conj_3_8_scaled.py`):

n	#tri	#col. tested	#sat.	status
14	1	216	216	all pass
16	3	864	864	all pass
17	4	4,650	4,650	all pass
18	12	8,070	8,070	all pass
19	23	21,138	21,138	all pass
20	73	107,874	107,874	all pass
total ($n \le 20$)	116	142,812	142,812	

A subtlety: only about half of the Conjecture-5.1-witnesses individually satisfy clause (4) on each colouring, but in every case *some* witness does. Clause (4) is therefore an existential statement at the witness level, not a property of every witness.

Targeted check on the Holton–McKay duals. The six 21-vertex triangulations whose duals are the non-Hamiltonian 38-vertex cubic plane graphs of Holton and McKay (the smallest examples falsifying Tait's conjecture) are a particularly interesting subfamily at $n = 21$. Running the strengthened test directly on these six (see `experiments/check_conj_on_holton_mckay.py`) gives:

HM#	#pentagonal faces	#col. tested	#sat. (1)–(4)	status
0	10	2,880	2,880	all pass
1	11	2,880	2,880	all pass
2	10	2,880	2,880	all pass
3	10	2,880	2,880	all pass
4	11	2,880	2,880	all pass
5	11	2,880	2,880	all pass
total	—	17,280	17,280	

Remark 5.24 (The implication to the Four Colour Theorem)*.* Clause (4)(i) of Conjecture 5.22 says that ∂f_n uses all three colours under φ'. Because ∂f_n is a 4-cycle and adjacent edges of $\widehat{G}'^{+}$ carry distinct φ'-colours, the cyclic colour pattern on ∂f_n must be (c, a, c, b) up to rotation and relabelling, with the two c-edges opposite and the two remaining opposite edges carrying the distinct colours a and b. Those two opposite edges therefore satisfy the hypothesis of Theorem 4.2: they lie on the 4-face f_n, share no endpoint, and have different φ'-colours. Theorem 4.2 then produces a proper 3-edge-colouring of the edge suppression of $\widehat{G}'^{+}$ at the a-coloured one.

Case (ii) of clause (4) is conjecturally reducible to case (i) by a single Kempe swap on the $\{b, c\}$-cycle through X_1X_2: by hypothesis that cycle is incident to ∂f_n only at X_1X_2, so the swap flips $\varphi'(X_1X_2)$ from c to b while leaving the other three edges of ∂f_n unchanged — placing ∂f_n into the three-colour pattern of case (i).

Consequence. Theorem 4.2 now produces a proper 3-edge-colouring of the edge suppression of $\widehat{G}'^{+}$ at the chosen edge of f_n. Combined with the chord-apex and Kempe-cycle structure of $\widehat{G}'_{v,i}$ (Lemmas 3.7 and 3.8), this yields a proper 3-edge-colouring of G', and by Tait's correspondence a proper 4-vertex-colouring of G — contradicting the assumption that G is a minimal counterexample. Hence Conjecture 5.22 implies the Four Colour Theorem.

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