

S_3 -orbit decomposition of $S_1 \cap S_2$

Do structurally different tires share the same canonical orbits?

The question

Step 2 (`tire_fiber_step2.tex`) reported $|S_1 \cap S_2|$ for 23 adjacent-tire pairs and found all 23 compatible. A follow-up question is whether structurally different (T_1, T_2) pairs that share a cycle-length $|\gamma| = k$ have intersections with the *same orbit structure* — i.e. whether a canonical pattern like the rainbow (a, b, c, b, c, a) persists when we vary T_1 and T_2 , or whether each pair gives its own pair-specific orbit.

Script: `experiments/orbit_decomposition.py`; raw output: `experiments/orbit_decomposition_data.txt`.

S_3 -closure: a structural sanity check

Permuting the three colors is a symmetry of proper edge 3-coloring, so for any fixed tire T , both S_1 and S_2 must be closed under the diagonal S_3 action on $\{1, 2, 3\}^k$. Hence so is $S_1 \cap S_2$.

Observation (S_3 -closure). *In every one of the 23 pairs, $S_1 \cap S_2$ is closed under the S_3 color action. This is structural rather than coincidental.*

Orbit size distribution

For $\sigma \in \{1, 2, 3\}^k$ with 3 distinct color-values, the S_3 orbit has size exactly 6. For σ using 2 distinct color values, the orbit has size 6 (since S_3 acts on the $\binom{3}{2} \cdot 2$ ways of placing them). For σ constant (one color), the orbit has size 3.

Observation (Uniform orbit sizes). *Across all 23 pairs, every S_3 -orbit in $S_1 \cap S_2$ has size exactly 6, with one single exception: the constant orbit $\{(1, \dots, 1), (2, \dots, 2), (3, \dots, 3)\}$ of size 3, which appears only in the case $\gamma = 4$, $T_1 = T_2 = (4, -, \text{SR})$.*

So intersection sizes are essentially $6 \cdot (\text{number of orbits})$ in all but one case.

The rainbow orbit reappears across structurally different pairs

Combining S_3 color action with cyclic rotation of γ gives a larger symmetry group; the combined orbit of $(1, 2, 3, 2, 3, 1)$ has size 36. This single combined orbit appears in three different (T_1, T_2) pairs at $\gamma = 6$:

γ	T_1	T_2
6	$(6, (0, 3), \text{SP})$	$(6, (0, 3), \text{SP})$
6	$(6, (0, 3), \text{SP})$	$(6, (0, 2)(3, 5), \text{SP})$
6	$(6, (0, 3), \text{SP})$	$(3, -, \text{SR})$

All three have $T_1 = (6, (0, 3), \text{SP})$ — the side with the *antipodal* chord. The three different T_2 structures range from another antipodal-chord SP tire to a chord-less SP tire to a chordless SR tire. In every case the rainbow orbit is part of (or all of) the intersection.

Observation (Rainbow orbit is T_1 -driven at $\gamma = 6$). *At $\gamma = 6$, the rainbow combined orbit (a, b, c, b, c, a) appears iff $T_1 = (6, (0, 3), \text{SP})$ — the antipodal-chord SP tire. The two-chord SP tire $T_1 = (6, (0, 2)(3, 5), \text{SP})$ yields different orbits, never the rainbow. So the rainbow pattern is associated with antipodal O-chord topology rather than with the pair (T_1, T_2) as a whole.*

Most-shared canonical orbits

Other canonical combined orbits appear across many structurally different pairs. Top entries:

$ \gamma $	canonical rep	# distinct pairs	combined-orbit size
4	(1, 2, 1, 3)	7	12
4	(1, 2, 1, 2)	6	6
4	(1, 2, 2, 1)	5	12
4	(1, 2, 2, 3)	5	24
3	(1, 2, 3)	4	6
5	(1, 2, 1, 2, 3)	3	30
6	(1, 2, 3, 2, 3, 1) (rainbow)	3	36

Observation (Universal orbits dominate). *At each cycle length k , the most-shared canonical orbit appears in roughly 30–40% of tested pairs at that k . These “universal” orbits are typically of the form (a, b, a, c) or (a, b, a, b) at $\gamma = 4$, (a, b, c) at $\gamma = 3$, etc. They survive across chord placements and across SR/SP model choices.*

What this means for chain pigeonhole

If $S_1 \cap S_2$ is always S_3 -closed (Obs.) and always contains at least one full S_3 -orbit (which, given size 6 for non-constant orbits, is essentially the same as saying $|S_1 \cap S_2| > 0$), then *chain pigeonhole at a single shared cycle is not just a counting fact but a structural one*: the intersection respects color symmetry, never collapses to a thin exotic subset, and contains canonical orbits that recur across structurally varied pairs.

This is a real upgrade on the step-2 data:

- Step 2 only reported $|S_1 \cap S_2| > 0$; it could in principle be a single weird configuration with no symmetry.
- The orbit decomposition shows the intersections always have the full S_3 -symmetric shape, with at least one orbit of size 6 (or 3 in the trivial case).
- Several canonical orbits recur across structurally different pairs — evidence that chain compatibility is detecting something structural about γ and the tire-pair topology, not a coincidence of one specific configuration.

Conjecture suggested by the data

Observation (Antipodal-chord rainbow conjecture). *Let $T = (m, (0, m/2), \text{SP})$ be a Steiner-poor tire whose inner outerplanar graph O is a cycle of length m together with a single antipodal chord (so m is even). Conjecture: the projection support $\pi_D(\mathcal{C}(T))$ on the $|\gamma| = m$ inner-side spokes always contains the combined orbit*

$$\text{Orbit}\big((a, b, c, b, c, \dots, b, c, a)\big)$$

under $S_3 \times C_m$ (color permutation \times cyclic rotation), where the pattern has length m and the a -positions are exactly the two chord endpoints, with b and c alternating elsewhere. At $m = 6$ this is the rainbow orbit of size 36 that Obs. witnessed.

If true, this is a uniform structural property of the antipodal-chord SP tire, independent of the outer boundary length. The chain pigeonhole step at $|\gamma| = m$ on such a tire reduces to “ π_U of the other tire intersects this fixed orbit,” a much smaller compatibility claim.

Direct test. At $m = 4$ ($\theta(1, 2, 2) = K_4 - e$) the antipodal-chord SP tire’s π_D support has size 36 at $|\gamma| = 4$ (`tire_fiber_chords.tex`, row “(4,4) chord (0,2)”), and the conjectured orbit $(a, b, c, b) \cdot S_3 \times C_4$ has size 24. Confirming the conjecture at $m = 4$ amounts to checking that this 24-element subset lies inside the 36-element support; this is mechanical.

Why antipodal? In the planar dual picture, the antipodal chord of O corresponds to the dual edge of a single “maximally-separating” chord in the tire’s inner outerplanar graph: it splits π_1 of the annulus most symmetrically. Any reasonable proof of the conjecture would have to exploit this symmetry — e.g. via reflection invariance on the chord axis.

Caveats

1. Still empirical for $k \leq 6$, chord counts ≤ 2 , and the 23 pairs from step 2.
2. “Same combined orbit” is taken modulo $S_3 \times C_k$ (color permutation \times cyclic rotation of γ). Reflection of γ is *not* quotiented out — some orbits would coincide if it were.
3. The rainbow orbit’s persistence is tied to the antipodal chord; a structural proof would need to show that the antipodal-chord SP tire’s projection support π_D always contains the rainbow orbit, independently of the outer tire.