

# HUMANS SUFFICE: A NOVEL PROOF OF THE FOUR COLOR THEOREM

ERIC BAUERFELD

*Dedicated to all who value more than machines*

ABSTRACT.

## 1. KEMPE'S PROOF (VALID PORTION)

**1.1. Setup.** Kempe's strategy, published in 1879, follows a *minimal counterexample* argument. Suppose, for contradiction, that there exists a planar graph requiring 5 colors. Among all such graphs, let  $G$  be one with the fewest vertices. Then every planar graph with fewer vertices than  $G$  is 4-colorable, but  $G$  itself is not.

### 1.2. Every Planar Graph Has a Vertex of Degree at Most 5.

**Lemma 1.1.** *Every planar graph has at least one vertex of degree  $\leq 5$ .*

*Proof.* Let  $G$  be a connected planar graph with  $V$  vertices,  $E$  edges, and  $F$  faces. By Euler's formula,

$$V - E + F = 2.$$

Since every face is bounded by at least 3 edges and each edge borders at most 2 faces, we have  $2E \geq 3F$ , hence  $F \leq \frac{2E}{3}$ . Substituting into Euler's formula:

$$V - E + \frac{2E}{3} \geq 2 \implies E \leq 3V - 6.$$

If every vertex had degree  $\geq 6$ , then  $2E \geq 6V$ , so  $E \geq 3V$ , contradicting  $E \leq 3V - 6$ . Therefore at least one vertex has degree  $\leq 5$ .  $\square$

Since  $G$  is a minimal counterexample, it must contain a vertex  $v$  of degree at most 5. Kempe argued by cases on the degree of  $v$ .

**1.3. Cases of Degree at Most 3.** Suppose  $v$  has degree  $\leq 3$ . Remove  $v$  from  $G$  to obtain the graph  $G - v$ . Since  $G - v$  has fewer vertices than  $G$ , it is 4-colorable by minimality. Fix such a 4-coloring. Now reinsert  $v$ : its at most 3 neighbors occupy at most 3 of the 4 colors, so at least one color remains available for  $v$ . This yields a valid 4-coloring of  $G$ , a contradiction.

---

2020 *Mathematics Subject Classification.* Primary .

*Key words and phrases.* Discrete , Four Color Theorem,

Dr. Joe Fields.

Holepunch.

**1.4. Case of Degree 4.** Suppose  $v$  has degree exactly 4 with neighbors  $a, b, c, d$  appearing in cyclic order around  $v$  in the planar embedding. Remove  $v$  and 4-color  $G - v$  by minimality. If the four neighbors do not all receive distinct colors, then at least one color is unused among them, and we may assign that color to  $v$ , giving a contradiction.

So assume  $a, b, c, d$  receive all four distinct colors; call them 1, 2, 3, 4 respectively. Define a *Kempe chain* to be a maximal connected subgraph whose vertices are colored with exactly two specified colors.

Consider the Kempe chain  $K_{13}$  containing  $a$  (using colors 1 and 3).

- **Case 1:**  $c$  is not in  $K_{13}$ . Swap colors 1 and 3 throughout  $K_{13}$ . This is still a valid coloring of  $G - v$ , and now  $a$  receives color 3, so color 1 is free for  $v$ .
- **Case 2:**  $c$  is in  $K_{13}$ . Then there is a path of alternating colors 1 and 3 from  $a$  to  $c$  in the planar embedding. Because  $a$  and  $c$  alternate around  $v$  with  $b$  and  $d$ , this path separates  $b$  from  $d$  in the plane. Therefore  $b$  and  $d$  lie in different Kempe chains for colors 2 and 4. Swap colors 2 and 4 in the chain containing  $b$ ; now  $b$  receives color 4, freeing color 2 for  $v$ .

In both cases we obtain a valid 4-coloring of  $G$ , a contradiction.

## 2. RESOLUTION OF DEGREE 5 CASE

**2.1. At Least 12 Vertices of Degree 5.** Since we have already handled every vertex of degree  $\leq 4$ , we may assume without loss of generality that the minimal counterexample  $G$  has minimum degree 5. We now show that  $G$  must contain at least 12 vertices of degree exactly 5.

**Lemma 2.1.** *If  $G$  is a planar graph with minimum degree 5, then  $G$  contains at least 12 vertices of degree exactly 5.*

*Proof.* For each  $k \geq 5$ , let  $n_k$  denote the number of vertices of degree exactly  $k$  in  $G$ . Since every vertex has degree  $\geq 5$ ,

$$V = \sum_{k \geq 5} n_k, \quad 2E = \sum_{k \geq 5} k n_k.$$

Using  $E \leq 3V - 6$ , we obtain  $2E \leq 6V - 12$ , so

$$\sum_{k \geq 5} k n_k \leq 6 \sum_{k \geq 5} n_k - 12.$$

Moving all terms to the right-hand side gives

$$12 \leq 6 \sum_{k \geq 5} n_k - \sum_{k \geq 5} k n_k = \sum_{k \geq 5} (6 - k) n_k.$$

We now expand this sum by separating the  $k = 5$ ,  $k = 6$ , and  $k \geq 7$  terms:

$$\sum_{k \geq 5} (6 - k) n_k = \underbrace{(6 - 5) n_5}_{= n_5} + \underbrace{(6 - 6) n_6}_{= 0} + \sum_{k \geq 7} (6 - k) n_k = n_5 - \sum_{k \geq 7} (k - 6) n_k.$$

Therefore  $12 \leq n_5 - \sum_{k \geq 7} (k - 6) n_k$ , which rearranges to

$$n_5 \geq 12 + \sum_{k \geq 7} (k - 6) n_k.$$

Since  $k - 6 \geq 1 > 0$  and  $n_k \geq 0$  for all  $k \geq 7$ , each term in the sum is non-negative, so  $\sum_{k \geq 7} (k - 6) n_k \geq 0$  and thus  $n_5 \geq 12$ .  $\square$

## 2.2. Two Non-Adjacent Vertices of Degree 5.

**Lemma 2.2.** *If  $G$  is a planar graph with minimum degree 5, then  $G$  contains two non-adjacent vertices of degree exactly 5.*

*Proof.* By the previous lemma,  $G$  contains at least 12 vertices of degree exactly 5. Let  $S$  denote the set of all degree-5 vertices, so  $|S| \geq 12$ . Suppose for contradiction that every two vertices in  $S$  are adjacent, i.e.,  $S$  induces a clique in  $G$ . Then every vertex  $v \in S$  is adjacent to all other  $|S| - 1 \geq 11$  vertices of  $S$ . But  $\deg(v) = 5$ , so  $v$  has at most 5 neighbors in total. Since  $11 > 5$ , this is a contradiction. Therefore  $S$  does not form a clique, and there exist two vertices  $v_1, v_2 \in S$  that are non-adjacent.  $\square$

**2.3. The Reduced Subgraph  $G'$ .** By the previous lemma, we may fix two non-adjacent vertices  $v_0, v_1 \in G$  each of degree 5. Define

$$G' = G - \{v_0, v_1\},$$

as the subgraph of  $G$  obtained by deleting  $v_0$  and  $v_1$  together with all edges incident to either vertex. Since  $G'$  has strictly fewer vertices than  $G$ , the minimality of  $G$  guarantees that  $G'$  admits a proper 4-coloring  $\phi : V(G') \rightarrow \{1, 2, 3, 4\}$ . Because  $v_0$  and  $v_1$  are non-adjacent in  $G$ , neither is a neighbor of the other, so each retains all 5 of its neighbors in  $G'$ . Let  $N(v_i)$  denote the set of neighbors of  $v_i$  in  $G$ ; then  $N(v_0)$  and  $N(v_1)$  are each sets of 5 vertices, all of which lie in  $G'$  and are assigned colors by  $\phi$ . Note also that  $N(v_0)$  and  $N(v_1)$  may overlap. Let  $\Phi$  be the set of all possible 4-colorings of  $G'$ .

## 2.4. Not All Colorings in $\Phi$ Saturate Both Neighborhoods.

**Lemma 2.3.** *It is not possible that every  $\phi \in \Phi$  satisfies both  $|\phi(N(v_0))| = 4$  and  $|\phi(N(v_1))| = 4$ .*

*Proof.* We show the stronger fact that there exists  $\phi \in \Phi$  with  $|\phi(N(v_0))| \leq 3$ .

Since  $v_1$  is a single vertex,  $G - \{v_1\}$  is a planar graph on  $V - 1$  vertices. Because  $V - 1 < V$  and  $G$  is a minimal counterexample,  $G - \{v_1\}$  is 4-colorable. Fix a proper 4-coloring  $\psi$  of  $G - \{v_1\}$ . This coloring assigns a color to every vertex of  $G'$  and also to  $v_0$ .

Let  $\phi = \psi|_{V(G')}$  be the restriction of  $\psi$  to  $G'$ . Since  $\psi$  is a proper coloring of  $G - \{v_1\} \supseteq G'$ , the restriction  $\phi$  is a proper 4-coloring of  $G'$ , so  $\phi \in \Phi$ .

Because  $\psi$  is a proper coloring of  $G - \{v_1\}$  and  $v_0 \in V(G - \{v_1\})$ , the color  $\psi(v_0)$  differs from the color of every neighbor of  $v_0$ . In particular,  $\psi(v_0) \notin \phi(N(v_0))$ . Since  $\psi(v_0) \in \{1, 2, 3, 4\}$  and  $\psi(v_0)$  does not appear in  $\phi(N(v_0))$ , it follows that  $|\phi(N(v_0))| \leq 3$ . Therefore not every coloring in  $\Phi$  uses all 4 colors on  $N(v_0)$ , and in particular it is impossible that every  $\phi \in \Phi$  saturates both  $N(v_0)$  and  $N(v_1)$  with all 4 colors.  $\square$

## 3. MERGED SUBGRAPHS OF $G'$

Let  $N(v_0) = \{a_0, a_1, a_2, a_3, a_4\}$  be the five neighbors of  $v_0$  in  $G'$ . For each pair of non-adjacent vertices  $a_i, a_j \in N(v_0)$  (i.e.,  $\{a_i, a_j\} \notin E(G')$ ), define the graph  $G'_{ij}$  to be the result of identifying  $a_i$  and  $a_j$  in  $G'$ : formally,  $G'_{ij}$  is obtained from  $G'$  by removing  $a_i$  and  $a_j$  and adding a new vertex  $a_{ij}$  adjacent to every vertex in  $N_{G'}(a_i) \cup N_{G'}(a_j)$ . Define the set of all such merged subgraphs by

$$\mathcal{M} = \{ G'_{ij} : a_i, a_j \in N(v_0), \{a_i, a_j\} \notin E(G') \}.$$

**3.1. Colorings of Merged Subgraphs Extend to  $G'$ .** For each  $G'_{ij} \in \mathcal{M}$ , let  $\Phi(G'_{ij})$  denote the set of proper 4-colorings of  $G'_{ij}$ . Each such coloring can be extended to a coloring of  $G'$  by assigning the color of the merged vertex  $a_{ij}$  back to both  $a_i$  and  $a_j$ . Formally, for  $\psi \in \Phi(G'_{ij})$ , define  $\tilde{\psi} : V(G') \rightarrow \{1, 2, 3, 4\}$  by

$$\tilde{\psi}(v) = \begin{cases} \psi(a_{ij}) & \text{if } v = a_i \text{ or } v = a_j, \\ \psi(v) & \text{otherwise.} \end{cases}$$

Let  $\tilde{\Phi}_{ij} = \{\tilde{\psi} : \psi \in \Phi(G'_{ij})\}$  denote the set of all such extensions.

**Lemma 3.1.** *For each  $G'_{ij} \in \mathcal{M}$ , we have  $\tilde{\Phi}_{ij} \subseteq \Phi$ .*

*Proof.* Let  $\psi \in \Phi(G'_{ij})$  and let  $\tilde{\psi}$  be its extension to  $G'$ . We verify that  $\tilde{\psi}$  is a proper coloring of  $G'$  by checking every edge  $\{u, w\} \in E(G')$ .

**Case 1: neither  $u$  nor  $w$  is  $a_i$  or  $a_j$ .** Then  $\{u, w\}$  is also an edge of  $G'_{ij}$ , and  $\tilde{\psi}(u) = \psi(u) \neq \psi(w) = \tilde{\psi}(w)$ .

**Case 2:  $u \in \{a_i, a_j\}$  and  $w \notin \{a_i, a_j\}$  (or vice versa).** Then  $w \in N_{G'}(a_i) \cup N_{G'}(a_j)$ , so  $w$  is adjacent to  $a_{ij}$  in  $G'_{ij}$ . Since  $\psi$  is proper,  $\psi(w) \neq \psi(a_{ij}) = \tilde{\psi}(u)$ .

**Case 3:  $u = a_i$  and  $w = a_j$ .** This case cannot occur, since  $a_i$  and  $a_j$  are non-adjacent in  $G'$  by the definition of  $\mathcal{M}$ .

In all cases the endpoints of every edge receive distinct colors, so  $\tilde{\psi} \in \Phi$ .  $\square$

**3.2. 4-Saturating Merged Colorings Extend to 4-Saturating Colorings of  $G'$ .** For each  $G'_{ij} \in \mathcal{M}$ , the vertices of  $G'_{ij}$  that would be adjacent to  $v_0$  if  $v_0$  were reinserted are exactly  $\{a_{ij}\} \cup (N(v_0) \setminus \{a_i, a_j\})$ ; call this set  $N_{ij}(v_0)$ . Define

$$\Phi_4 = \{\phi \in \Phi : |\phi(N(v_0))| = 4\}$$

to be the set of colorings of  $G'$  that use all 4 colors on  $N(v_0)$ , and for each  $G'_{ij} \in \mathcal{M}$  define

$$\Phi_4(G'_{ij}) = \{\psi \in \Phi(G'_{ij}) : |\psi(N_{ij}(v_0))| = 4\}$$

to be the colorings of  $G'_{ij}$  that use all 4 colors on the effective neighborhood of  $v_0$ .

**Lemma 3.2.** *For each  $G'_{ij} \in \mathcal{M}$ , the extensions of  $\Phi_4(G'_{ij})$  lie in  $\Phi_4$ , i.e.,  $\widetilde{\Phi_4(G'_{ij})} \subseteq \Phi_4$ .*

*Proof.* Let  $\psi \in \Phi_4(G'_{ij})$ , so  $|\psi(N_{ij}(v_0))| = 4$ . By definition of the extension,

$$\tilde{\psi}(N(v_0)) = \tilde{\psi}(\{a_i, a_j\} \cup (N(v_0) \setminus \{a_i, a_j\})) = \{\psi(a_{ij})\} \cup \psi(N(v_0) \setminus \{a_i, a_j\}) = \psi(N_{ij}(v_0)).$$

The first equality expands  $N(v_0)$ ; the second uses  $\tilde{\psi}(a_i) = \tilde{\psi}(a_j) = \psi(a_{ij})$  and  $\tilde{\psi}(u) = \psi(u)$  for  $u \notin \{a_i, a_j\}$ ; the third uses  $N_{ij}(v_0) = \{a_{ij}\} \cup (N(v_0) \setminus \{a_i, a_j\})$ . Therefore  $|\tilde{\psi}(N(v_0))| = |\psi(N_{ij}(v_0))| = 4$ . Since also  $\tilde{\psi} \in \Phi$  by the previous lemma, we conclude  $\tilde{\psi} \in \Phi_4$ .  $\square$

*Email address:* eric@dideric.is