PLANE DIAMOND COLORING

ERIC BAUERFELD

Abstract.

Notation

For a coloring $C : V(G) \to S$ and a color $c \in S$, we write $C^{-1}(c) = \{v \in V(G) : C(v) = c\}$ for the preimage of c under C, i.e., the color class of c.

1. Definitions

Definition 1.1. Let G be a graph and let $u \in V(G)$. The *distance partition* of G from u is the partition $\{L_0, L_1, L_2, \dots\}$ of $V(G)$ obtained by breadth-first search from u:

$$L_0 = \{u\}, \qquad L_{i+1} = \{v \in V(G) \setminus (L_0 \cup \dots \cup L_i) : v \text{ is adjacent to some } w \in L_i\}.$$

Equivalently, $L_i = \{v \in V(G) : d(v,u) = i\}$, where $d(v,u)$ denotes the graph distance between v and u in G. We call each L_i the *i-th level* of the partition.

Definition 1.2. Let G be a maximal planar graph. A *plane diamond coloring* of G is a proper 4-coloring C of G for which there exist a vertex $u \in V(G)$ and two distinct colors c_a, c_b such that, with respect to the distance partition $\{L_0, L_1, L_2, \dots\}$ of G from u,

$$C^{-1}(c_a) \subseteq \bigcup_{i \text{ even}} L_i \qquad \text{and} \qquad C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i.$$

2. Results

Remark 2.1. Definition 1.2 imposes a structural condition on 4-colorings of maximal planar graphs strictly stronger than the conclusion of the Four Color Theorem [1, 2]: it requires not merely the existence of a proper 4-coloring, but the existence of a proper 4-coloring together with a root u such that two of the four color classes are separated by the parity of the BFS layering from u.

Conjecture 2.2. *Every maximal planar graph G has a plane diamond coloring.*

Theorem 2.3. *The preceding conjecture is false. Moreover, the smallest counterexample has order* 13*, and is unique up to isomorphism among triangulations of order at most* 13*.*

Proof. Let G be the maximal planar graph on 13 vertices with graph6 string[1]

`LQhO?SFIwnTYTi`

2010 *Mathematics Subject Classification.* Primary .

[1] We use the standard graph6 encoding of McKay; see [3].

shown in Figure 1. Equivalently, G has edge set

$$\{0,2\},\{0,4\},\{0,11\},\{1,3\},\{1,5\},\{1,12\},\{2,4\},\{2,9\},\{2,11\},$$
$$\{3,5\},\{3,10\},\{3,12\},\{4,7\},\{4,9\},\{4,11\},\{5,8\},\{5,10\},\{5,12\},$$
$$\{6,7\},\{6,8\},\{6,9\},\{6,10\},\{6,11\},\{6,12\},\{7,8\},\{7,9\},\{7,10\},$$
$$\{7,11\},\{8,9\},\{8,10\},\{8,12\},\{9,11\},\{10,12\}.$$

We have $|V(G)| = 13$ and $|E(G)| = 33 = 3 \cdot 13 - 6$, so G is a triangulation.

By Definition 1.2, it suffices to show that for every root $u \in V(G)$, no proper 4-coloring C of G admits two distinct colors c_a, c_b with $C^{-1}(c_a)$ contained in the union of even-indexed levels and $C^{-1}(c_b)$ contained in the union of odd-indexed levels of the distance partition from u.

For a fixed root u, the existence of such a triple (C, c_a, c_b) is equivalent to 4-colorability of the auxiliary graph H_u obtained from G by adjoining two new vertices α, β, joining α to every vertex in odd-indexed levels, joining β to every vertex in even-indexed levels, and adding the edge $\{\alpha, \beta\}$. Indeed, in any proper 4-coloring of H_u the colors of α and β are distinct and absent from the odd-parity and even-parity layers of G respectively, yielding $c_a := C(\alpha)$ and $c_b := C(\beta)$. Conversely, given a 4-coloring satisfying the parity-separation condition, setting $C(\alpha) := c_a$ and $C(\beta) := c_b$ extends it to a proper 4-coloring of H_u.

A direct computation (using `Sage`'s `chromatic_number`) verifies that $\chi(H_u) > 4$ for every $u \in V(G)$, so G admits no plane diamond coloring.

For minimality and uniqueness, we exhaustively enumerated every maximal planar graph of order at most 13 using `Sage`'s `graphs.planar_graphs` generator (with `minimum_connectivity=3` and `maximum_face_size=3`). The numbers of triangulations $1, 1, 2, 5, 14, 50, 233, 1249, 7595, 49566$ at orders $4, 5, \ldots, 13$ respectively (matching OEIS A000109) were each tested for the existence of a plane diamond coloring, and exactly one — the graph G above, occurring at order 13 — was found to lack one. $\square$

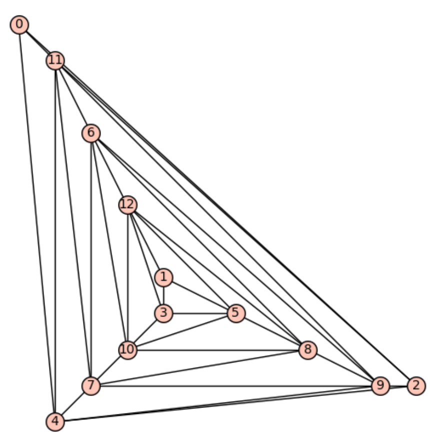

FIGURE 1. The unique smallest maximal planar graph with no plane diamond coloring; it has 13 vertices and degree sequence $(6, 6, 6, 6, 6, 6, 6, 5, 5, 4, 4, 3, 3)$.

Conjecture 2.4. *Every maximal planar graph G of minimum degree at least 5 has a plane diamond coloring.*

Remark 2.5. We have verified Conjecture 2.4 computationally for all maximal planar graphs of minimum degree at least 5 and order at most N, by exhaustive enumeration via `Sage`'s `graphs.planar_graphs` generator and the auxiliary-graph reduction described in the proof of Theorem 2.3. No counterexample has been found.

References

[1] K. Appel and W. Haken, *Every planar map is four colorable*, Illinois Journal of Mathematics, vol. 21, no. 3, pp. 429–567, 1977.

[2] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, *The four-colour theorem*, Journal of Combinatorial Theory, Series B, vol. 70, no. 1, pp. 2–44, 1997.

[3] B. D. McKay, *Description of graph6, sparse6 and digraph6 encodings*, `https://users.cecs.anu.edu.au/~bdm/data/formats.txt`.