COLORING NESTED TIRE GRAPHS

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Abstract.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

We study the structure such a minimal counterexample would have to exhibit through the lens of *nested level duals*. Fixing a level source S in G endows the dual G' with a Breadth-First-Search–derived labelling, the dual depth of Definition 1.4, and the level structure of G organises G' into a family of nested cycles carrying these labels. Our aim is to express the obstruction to a 3-edge-colouring of G' as conditions on this nested labelled-cycle structure.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \mathrm{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

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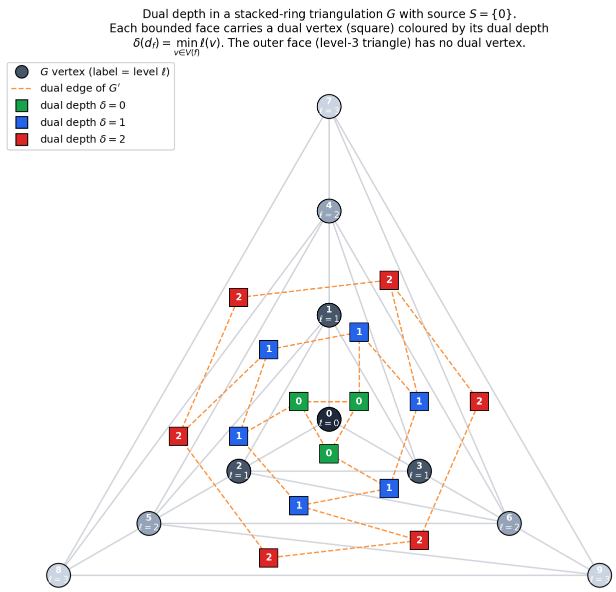

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.5 (Tire graph)**.** A *tire graph* consists of a plane graph T together with two *boundary parts* $B_{\text{out}}, B_{\text{in}} \subseteq T$ and an *inner outerplanar graph* $O \subseteq T$, where each of B_{out} and the outer-face boundary B_{in} of O is either

- a simple cycle of length ≥ 3, or
- a single vertex (a *degenerate* boundary),

with at most one of $B_{\text{out}}, B_{\text{in}}$ degenerate, and $V(B_{\text{out}}) \cap V(O) = \emptyset$. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \qquad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T, the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R. The region R is a closed annulus when both B_{out} and B_{in} are cycles, and a closed disk when exactly one of them is a single vertex.

We call B_{out} the *outer boundary*, O the *inner outerplanar graph*, and B_{in} the *inner boundary* of T. A tire graph in which B_{out} (respectively B_{in}) is a single vertex is said to have a *degenerate outer (respectively inner) boundary*; in either case T is a triangulated closed disk with that vertex as apex.

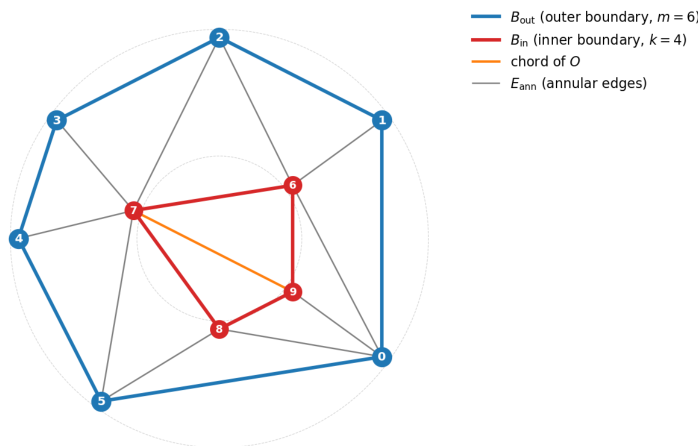

FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \ldots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \ldots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7\text{–}9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Remark 1.6. Let $m = |V(B_{\text{out}})|$ and $k = |V(B_{\text{in}})|$. By Euler's formula on the annular (resp. disk) region R, the tire graph has $m+k$ triangular faces inside R and $|E_{\text{ann}}| = m + k$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(m, k) = 1$), there are $m+k-1$ triangular faces and $|E_{\text{ann}}| = m + k - 1$.

Lemma 1.7 (Tire-component lemma)**.** *Let G be a maximal planar graph with fixed embedding Π_G and let $S \subseteq V(G)$ be a level source. For $d \geq 0$, let*

$$G'_d \;:=\; G'\big[\,\{d_f \in V(G') : \delta_G(d_f) = d\}\,\big]$$

be the inner-dual subgraph on dual vertices of dual depth d, and let C' be a connected component of G'_d. Write $F_{C'} := \{f : d_f \in V(C')\}$, $V_{C'} := \bigcup_{f \in F_{C'}} V(f)$, and $C := G[V_{C'}]$ with the embedding inherited from Π_G.

Then C, together with its inherited embedding, is a tire graph in the sense of Definition 1.5: the two boundary parts $\{B_{\text{out}}, B_{\text{in}}\}$ of C are the level-d subgraph $G[V_{C'} \cap L_d]$ and the level-$(d+1)$ subgraph $G[V_{C'} \cap L_{d+1}]$, in either order, and the triangular faces of C inside its closed boundary region are exactly the faces of G in $F_{C'}$.

Proof sketch. By Lemma 2.6 of [1] (whose argument, given for an outer-cycle source, extends verbatim to an arbitrary level source S by treating S as the depth-0 set), the subgraph $G[L_{d'}]$ is outerplanar for each $d' \geq 0$. Since subgraphs of outerplanar graphs are outerplanar, both $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are outerplanar.

Layer containment (a consequence of the bounded-step property of BFS on a triangulation) gives $V_{C'} \subseteq L_d \cup L_{d+1}$, so C has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$, and every face $f \in F_{C'}$ is a triangle with at least one vertex in L_d.

It remains to identify the boundary of $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$. Each edge on the topological boundary $\partial R_{C'}$ separates a face $f \in F_{C'}$ (depth d) from a face $f' \notin F_{C'}$. Such an f' has dual depth in $\{d-1, d+1\}$: if d, then f' shares an edge with a depth-d face but lies in a distinct component of G'_d, which contradicts the connectivity of C' together with the fact that adjacent depth-d dual vertices belong to the same component. A short case analysis on the level of the third vertex of f' shows that boundary edges with $\delta(f') = d-1$ have both endpoints in L_d, while those with $\delta(f') = d+1$ have both endpoints in L_{d+1}. Each connected component of $\partial R_{C'}$ is therefore monochromatic in level.

Since C' is connected and $R_{C'}$ is a connected planar 2-complex of triangles glued along edges, $\partial R_{C'}$ consists of either one closed walk (when $R_{C'}$ is a topological disk) or two closed walks (when $R_{C'}$ is a topological annulus). These walks are simple cycles in G on the respective level sets (with one cycle possibly degenerating to a single vertex of L_d or L_{d+1} at the endpoints of the BFS, giving the degenerate-boundary case of Definition 1.5). Together with the outerplanarity of $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ established above, these cycles serve as the two boundary parts of C, in either order, and the depth-d triangles in $F_{C'}$ tile the closed region between them. □

Remark 1.8. Either boundary part of C in Lemma 1.7 may be degenerate. For instance, at $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ — the degenerate boundary — and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G). Which of the two parts is B_{out} and which is B_{in} depends on the orientation of the inherited embedding (equivalently, on which side of C contains the rest of Π_G).

References

[1] E. Bauerfeld, *Plane Depth Sequencing*, manuscript (math-research repository), 2026.