Fiber distributions for tire face connectors with chords in O

What this is

A companion to `tire_fiber_data.tex`: that note enumerated fiber distributions in the spoke-only setting (no chords in the inner outerplanar graph O). This note investigates what happens when O has chords — specifically when the surrounding G does *not* introduce Steiner vertices inside B_{in}, so that each face of O in the tire's embedding remains a single face of G. Script: `experiments/tire_fiber_chords.py`. Output: `experiments/tire_fiber_chords_data.txt`.

The modelling choice that chords force on us

In the spoke-only enumeration (`tire_fiber_data.tex`) the inner-spoke part of $T'_{f'}$ was simply k pendant edges, one per B_{in} edge. That picture corresponds to a surrounding maximal planar graph G that *aggressively triangulates* the disk inside B_{in} using interior (Steiner) vertices not in $V(T)$, so each B_{in} edge gets its own dedicated inside-O sub-triangle.

When we add chords to O, there is a second axis of choice: does G sub-triangulate the resulting O-faces, or not? Two extreme models:

- **Steiner-rich (SR).** G adds whatever vertices and edges it needs inside each O-face so that every B_{in} edge has its own sub-triangle. The inner-spoke graph is k pendants *regardless of the chord set* — chords are invisible to $T'_{f'}$. This is the step-1 baseline.
- **Steiner-poor (SP).** G does not further sub-triangulate O-faces beyond what O itself provides. Each face of O becomes a single face of G, whose dual vertex has degree (in G', and hence in $T'_{f'}$) equal to the number of B_{in} edges on the face boundary. Edge 3-colourable only if *every O-face has at most* 3 B_{in} *edges.*

Real surrounding triangulations G live somewhere between these two, and the choice is not part of the tire structure itself — it is a property of how G embeds the tire. But the two extremes already say something useful about chord effects.

Side-by-side data

case	n	P_e	$\lvert\pi_D(\mathcal{C})\rvert/3^k$	$\lvert\pi_U(\mathcal{C})\rvert/3^m$	P_e	$\lvert\pi_D(\mathcal{C})\rvert/3^k$	$\lvert\pi_U(\mathcal{C})\rvert/3^m$
		Steiner-rich			**Steiner-poor**		
(4,4) no chord	8	258	81/81	81/81	0	0/81	0/81
(4,4) chord (0,2), faces 2+2	8	258	81/81	81/81	144	36/81	54/81
(5,5) no chord	10	1026	243/243	243/243	0	0/243	0/243
(5,5) chord (0,2), faces 2+3	10	1026	243/243	243/243	240	36/243	90/243
(5,5) chord (0,3), faces 3+2	10	1026	243/243	243/243	240	36/243	90/243
(6,6) no chord	12	4098	729/729	729/729	0	0/729	0/729
(6,6) chord (0,3) antipodal, 3+3	12	4098	729/729	729/729	408	36/729	90/729
(6,6) chords (0,2)(3,5), 2+2+2	12	4098	729/729	729/729	1536	216/729	456/729
(3,4) no chord	7	126	78/81	27/27	0	0/81	0/27
(3,4) chord (0,2)	7	126	78/81	27/27	48	36/81	24/27
(6,4) no chord	10	1026	81/81	549/729	0	0/81	0/729
(6,4) chord (0,2)	10	1026	81/81	549/729	480	36/81	342/729

(π_D projects σ onto the k inner-direction spoke positions on the dual cycle; π_U onto the m outer-direction positions. "Chord (a,b)" means $O = B_{\text{in}} \cup \{v_a v_b\}$.)

Observations

Observation (SP is empty without chords for $k > 3$)**.** *If O has no chord and $k \geq 4$, then O has a single inner face with k incident B_{in} edges. Under SP this is a single G'-vertex of degree k, and proper edge 3-colouring needs k distinct colours at that vertex, which is impossible for $k \geq 4$.* ***So the SP model forces chords once*** *$k \geq 4$: tires whose inner boundary is longer than 3 are only feasible if O already contains enough chords to break every face down to ≤ 3 B_{in} edges.*

This is the structural reason chord-aware modelling matters. Under SR, you can pretend chord sets do not exist; under SP, you cannot.

Observation (Adding chords *enlarges* the SP support)**.** *Comparing the two $(6,6)$ chord rows:*

- *Antipodal $(0,3)$ chord: face sizes 3+3, gives $P_e = 408$, π_D support $36/729 \approx 4.9\%$.*
- *Two chords $(0,2),(3,5)$: face sizes 2+2+2, gives $P_e = 1536$, π_D support $216/729 \approx 29.6\%$.*

Smaller faces = weaker constraints = more realisable configurations. This is the opposite of what one might expect at first glance ("more chords = more edges = more constraints") — each chord splits a hard constraint into easier ones.

Observation (π_D support depends only on chord structure, not on m)**.** *For chord set $\{(0,2)\}$ in $k = 4$, the π_D support is $36/81 = 4/9$ for every tested $m \in \{3,4,6\}$. This is structural: the chord constraint pins down which spoke-quadruples are realisable without involving the outer cycle at all. The π_U support, by contrast, varies with m.*

Observation (SP support shrinks the π_U support too)**.** *Even though chord constraints live on the inner side, they propagate to the outer side via the cycle. For $(m,k) = (4,4)$:*

- *SR: π_U saturates 3^4.*
- *SP with chord $(0,2)$: $\pi_U = 54/81 = 2/3$.*

The chord-induced constraints on D-position spokes propagate around the cycle through the proper-edge-colouring constraint, reducing the set of cycle colourings, hence reducing π_U as well.

Implications for the chain-pigeonhole step

In the spoke-only / SR step-1 analysis we observed that whenever $|B_{\text{out}}| \geq |B_{\text{in}}|$ the inner-side projection saturates the full ring-coloring universe 3^k, so chain compatibility on a shared cycle was trivially nonempty whenever at least one of the two adjacent tires had the longer non-shared boundary.

Under SP this clean saturation breaks down on the chord side:

- For a chorded tire, π_D never reaches 3^k — the chord constraints permanently rule out some inner-spoke configurations. The realisable set is small but non-empty (e.g., 36/81 at $k = 4$).
- The shrinkage is the same regardless of m on the inner (D) side, so making the outer boundary very long does *not* help recover saturation on the chord side.
- Two adjacent SP tires sharing a cycle γ both have shrunk projections; whether they intersect is now a real question. E.g., projecting π_D down to a 4/9-fraction of $\{1, 2, 3\}^k$ from each side: do the two 4/9-sets always meet?

That last question is the natural follow-up experiment (and is step 2 of the action items).

The structural fact worth stating

Observation (Steiner-poor chord-rule). *Under the SP model, a tire T is edge-3-colourable on its face connector $T'_{f'}$ iff its inner outerplanar graph O* already triangulates the inside of B_{in} up to faces of B_{in}- size ≤ 3. *Equivalently: O must contain enough chords that every O-face has at most 3 B_{in} edges on its boundary.*

For a level-source BFS decomposition with O chosen minimally (just B_{in} itself, no chords), every tire with $k \geq 4$ fails the SP edge-3-colouring test *locally.* Globally this is no contradiction — the surrounding maximal planar G does add Steiner vertices and edges inside B_{in}, making us closer to SR than to SP. But the SP model is a worst-case envelope on what purely local data implies, and shows that any structural argument that ignores the inside-B_{in} triangulation is incomplete.

Caveats / what we did not compute

1. **Only the two extremes.** Real G triangulations lie between SR (every B_{in} edge gets its own Steiner-sub-triangle) and SP (no further sub-triangulation at all). Intermediate cases (e.g., fan-triangulating one O-face from a $V(B_{\text{in}})$ vertex without adding Steiner points) are not enumerated here, though the data suggests they sit between.
2. **Outer side always SR.** We have always treated the outer side (B_{out}) as having no chord-induced structure — B_{out} is just a cycle in the tire definition, so this is correct, but the analogous question "what if the region outside B_{out} is poorly triangulated" is not addressed.
3. **$\Delta \leq 3$ on O ignored.** The chord configurations tested above include cases where chord endpoints have higher degree in T; the menagerie's $\Delta \leq 3$ constraint is not enforced. This is fine for the SP model in isolation but matters when composing with the menagerie counting formulas.
4. **Adjacency intersection (step 2).** We have only computed single-tire fiber distributions. The chain- pigeonhole step itself — intersecting π_D of one tire with π_U of the adjacent tire on a shared cycle — is deferred.