

PLANE DIAMOND COLORING

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ABSTRACT.

NOTATION

For a coloring $C : V(G) \rightarrow S$ and a color $c \in S$, we write $C^{-1}(c) = \{v \in V(G) : C(v) = c\}$ for the preimage of c under C , i.e., the color class of c .

1. MOTIVATION

Let G be a maximal planar graph. By the Four Color Theorem [1, 2], G admits a proper 4-coloring; the question that motivates this paper is whether such a coloring can always be exhibited via a particular structural construction, rather than as the output of an ad hoc case analysis.

The construction we have in mind proceeds as follows. Fix a root vertex $u \in V(G)$ and consider the BFS layering $\{L_0, L_1, L_2, \dots\}$ of G from u (Definition 2.1). Remove from G every edge whose two endpoints lie in the same layer L_i , producing a spanning subgraph $G^\diamond \subseteq G$ which we call the *diamond scaffold* of G relative to u (Definition 2.2). For any edge $\{x, y\} \in E(G)$ the BFS depths of x and y differ by at most 1, so every edge surviving in G^\diamond joins some level L_i to L_{i+1} ; in particular G^\diamond is bipartite, and the parity of the layer index supplies a canonical proper 2-coloring of G^\diamond using two colors, which we denote c_a and c_b .

To extend this 2-coloring of the scaffold to a proper 4-coloring of the original graph G , we must dispose of the edges discarded in passing from G to G^\diamond — namely the edges whose endpoints share a BFS layer. The natural strategy is to recolor a chosen subset of vertices using two new colors c_c, c_d , just enough to resolve the discarded same-layer conflicts while preserving the canonical $\{c_a, c_b\}$ -coloring on the remaining vertices. A *plane diamond coloring* of G (Definition 2.3) is precisely a proper 4-coloring of G obtained in this way: two of its color classes, $C^{-1}(c_a)$ and $C^{-1}(c_b)$, are confined to even-indexed and odd-indexed BFS layers respectively, so that they extend the canonical 2-coloring of the diamond scaffold G^\diamond relative to some root u .

This paper investigates which maximal planar graphs admit such a coloring. The Four Color Theorem guarantees a proper 4-coloring; the diamond coloring asks for one obeying the additional structural constraint above. Theorem 3.3 shows the constraint genuinely fails on some triangulations, with a unique smallest obstruction of order 13; Conjecture 3.4 asserts that the obstructions disappear once $\delta(G) \geq 5$.

2. DEFINITIONS

Definition 2.1. Let G be a graph and let $u \in V(G)$. The *distance partition* of G from u is the partition $\{L_0, L_1, L_2, \dots\}$ of $V(G)$ obtained by breadth-first search from u :

$$L_0 = \{u\}, \quad L_{i+1} = \{v \in V(G) \setminus (L_0 \cup \dots \cup L_i) : v \text{ is adjacent to some } w \in L_i\}.$$

Equivalently, $L_i = \{v \in V(G) : d(v, u) = i\}$, where $d(v, u)$ denotes the graph distance between v and u in G . We call each L_i the *i-th level* of the partition.

Definition 2.2. Let G be a maximal planar graph and let $u \in V(G)$, with distance partition $\{L_0, L_1, L_2, \dots\}$ from u . The *diamond scaffold* of G relative to u is the spanning subgraph $G^\diamond \subseteq G$ obtained from G by removing every edge $\{x, y\} \in E(G)$ such that $x, y \in L_i$ for some i .

Definition 2.3. Let G be a maximal planar graph. A *plane diamond coloring* of G is a proper 4-coloring C of G for which there exist a vertex $u \in V(G)$ and two distinct colors c_a, c_b such that, with respect to the distance partition $\{L_0, L_1, L_2, \dots\}$ of G from u ,

$$C^{-1}(c_a) \subseteq \bigcup_{i \text{ even}} L_i \quad \text{and} \quad C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i.$$

3. RESULTS

Remark 3.1. Definition 2.3 imposes a structural condition on 4-colorings of maximal planar graphs strictly stronger than the conclusion of the Four Color Theorem [1, 2]: it requires not merely the existence of a proper 4-coloring, but the existence of a proper 4-coloring together with a root u such that two of the four color classes are separated by the parity of the BFS layering from u .

Conjecture 3.2. *Every maximal planar graph G has a plane diamond coloring.*

Theorem 3.3. *The preceding conjecture is false. Moreover, the smallest counterexample has order 13, and is unique up to isomorphism among triangulations of order at most 13.*

Proof. Let G be the maximal planar graph on 13 vertices with graph6 string¹

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shown in Figure 1. Equivalently, G has edge set

$$\begin{aligned} &\{0, 2\}, \{0, 4\}, \{0, 11\}, \{1, 3\}, \{1, 5\}, \{1, 12\}, \{2, 4\}, \{2, 9\}, \{2, 11\}, \\ &\{3, 5\}, \{3, 10\}, \{3, 12\}, \{4, 7\}, \{4, 9\}, \{4, 11\}, \{5, 8\}, \{5, 10\}, \{5, 12\}, \\ &\{6, 7\}, \{6, 8\}, \{6, 9\}, \{6, 10\}, \{6, 11\}, \{6, 12\}, \{7, 8\}, \{7, 9\}, \{7, 10\}, \\ &\{7, 11\}, \{8, 9\}, \{8, 10\}, \{8, 12\}, \{9, 11\}, \{10, 12\}. \end{aligned}$$

We have $|V(G)| = 13$ and $|E(G)| = 33 = 3 \cdot 13 - 6$, so G is a triangulation.

By Definition 2.3, it suffices to show that for every root $u \in V(G)$, no proper 4-coloring C of G admits two distinct colors c_a, c_b with $C^{-1}(c_a)$ contained in the union of even-indexed levels and $C^{-1}(c_b)$ contained in the union of odd-indexed levels of the distance partition from u .

¹We use the standard graph6 encoding of McKay; see [3].

For a fixed root u , the existence of such a triple (C, c_a, c_b) is equivalent to 4-colorability of the auxiliary graph H_u obtained from G by adjoining two new vertices α, β , joining α to every vertex in odd-indexed levels, joining β to every vertex in even-indexed levels, and adding the edge $\{\alpha, \beta\}$. Indeed, in any proper 4-coloring of H_u the colors of α and β are distinct and absent from the odd-parity and even-parity layers of G respectively, yielding $c_a := C(\alpha)$ and $c_b := C(\beta)$. Conversely, given a 4-coloring satisfying the parity-separation condition, setting $C(\alpha) := c_a$ and $C(\beta) := c_b$ extends it to a proper 4-coloring of H_u .

A direct computation (using Sage's `chromatic_number`) verifies that $\chi(H_u) > 4$ for every $u \in V(G)$, so G admits no plane diamond coloring.

For minimality and uniqueness, we exhaustively enumerated every maximal planar graph of order at most 13 using Sage's `graphs.planar_graphs` generator (with `minimum_connectivity=3` and `maximum_face_size=3`). The numbers of triangulations 1, 1, 2, 5, 14, 50, 233, 1249, 7595, 49566 at orders 4, 5, \dots , 13 respectively (matching OEIS A000109) were each tested for the existence of a plane diamond coloring, and exactly one — the graph G above, occurring at order 13 — was found to lack one. \square

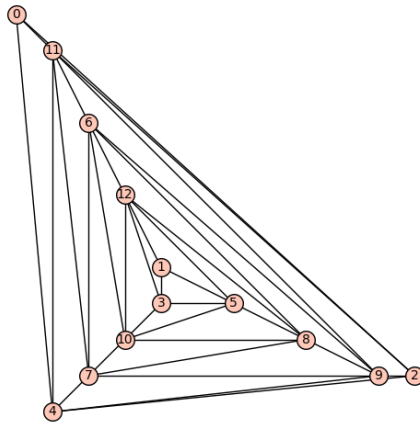


FIGURE 1. The unique smallest maximal planar graph with no plane diamond coloring; it has 13 vertices and degree sequence $(6, 6, 6, 6, 6, 6, 6, 5, 5, 4, 4, 3, 3)$.

Conjecture 3.4. *Every maximal planar graph G of minimum degree at least 5 has a plane diamond coloring.*

Remark 3.5. We have verified Conjecture 3.4 computationally for all maximal planar graphs of minimum degree at least 5 and order at most N , by exhaustive enumeration via Sage's `graphs.planar_graphs` generator and the auxiliary-graph reduction described in the proof of Theorem 3.3. No counterexample has been found.

REFERENCES

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- [3] B. D. McKay, *Description of graph6, sparse6 and digraph6 encodings*, <https://users.cecs.anu.edu.au/~bdm/data/formats.txt>.