PLANE DIAMOND COLORING

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Abstract.

Notation

For a coloring $C : V(G) \to S$ and a color $c \in S$, we write $C^{-1}(c) = \{v \in V(G) : C(v) = c\}$ for the preimage of c under C, i.e., the color class of c.

1. Definitions

Definition 1.1. Let G be a graph and let $u \in V(G)$. The *distance partition* of G from u is the partition $\{L_0, L_1, L_2, \dots\}$ of $V(G)$ obtained by breadth-first search from u:

$$L_0 = \{u\}, \qquad L_{i+1} = \{v \in V(G) \setminus (L_0 \cup \cdots \cup L_i) : v \text{ is adjacent to some } w \in L_i\}.$$

Equivalently, $L_i = \{v \in V(G) : d(v,u) = i\}$, where $d(v,u)$ denotes the graph distance between v and u in G. We call each L_i the *i-th level* of the partition.

Definition 1.2. Let G be a maximal planar graph with a plane embedding, and let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from some $u \in V(G)$. The *diamond scaffold* of G relative to u is the spanning subgraph $G^\diamond \subseteq G$ obtained by removing every edge $\{x, y\} \in E(G)$ such that $x, y \in L_i$ for some i.

Definition 1.3. Let G be a maximal planar graph. A *plane diamond coloring* of G is a proper 4-coloring C of G such that there exist two colors c_a, c_b and a diamond scaffold $G^\diamond$ of G with a proper 2-coloring $C^\diamond : V(G) \to \{c_a, c_b\}$ satisfying

$$\begin{aligned} C^\diamond(v) &= c_a \quad \text{for every } v \in C^{-1}(c_a), \\ C^\diamond(v) &= c_b \quad \text{for every } v \in C^{-1}(c_b). \end{aligned}$$

2. Results

Theorem 2.1. *The diamond scaffold of any maximal planar graph G is 2-colorable.*

Proof. Let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from the chosen vertex u, and let $G^\diamond$ be the resulting diamond scaffold. We show $G^\diamond$ is bipartite by exhibiting a proper 2-coloring.

For any edge $\{x, y\} \in E(G)$, the depths of x and y differ by at most 1: if $x \in L_i$, then prepending the edge $\{y, x\}$ to a shortest path from x to u gives a walk of length $i+1$ from y to u, so $y \in L_j$ for some $j \leq i+1$, and symmetrically $i \leq j+1$. Hence $|i - j| \leq 1$.

By construction, $G^\diamond$ contains no edge with both endpoints in the same level L_i. Combined with the bound above, every edge of $G^\diamond$ joins some L_i to L_{i+1}. Color

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each vertex $v \in L_i$ by the parity of i. Every edge of $G^\diamond$ connects vertices of opposite parity, so this is a proper 2-coloring. □

Lemma 2.2. *The diamond scaffold $G^\diamond$ of a maximal planar graph G relative to u is connected.*

Proof. Let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from u. We show by induction on i that every vertex of L_i is connected to u in $G^\diamond$. The base case $i = 0$ is immediate, since $L_0 = \{u\}$. For $i \geq 1$, let $v \in L_i$. By definition of L_i, there is a shortest path from v to u of length i in G, whose penultimate vertex w lies in L_{i-1}. The edge $\{v, w\}$ joins L_i to L_{i-1}, hence is not a level edge, hence belongs to $G^\diamond$. By the inductive hypothesis w is connected to u in $G^\diamond$, so v is as well. □

Proposition 2.3. *A maximal planar graph G has a plane diamond coloring if and only if there exist a proper 4-coloring C of G, a vertex $u \in V(G)$, and two distinct colors c_a, c_b such that, with respect to the distance partition $\{L_0, L_1, L_2, \dots\}$ of G from u,*

$$C^{-1}(c_a) \subseteq \bigcup_{i \text{ even}} L_i \qquad \text{and} \qquad C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i.$$

Proof. Since $G^\diamond$ is connected and bipartite (Theorem 2.1 and Lemma 2.2), its proper 2-coloring is unique up to swapping the two colors, and is given by the parity of level. Hence a proper 2-coloring $C^\diamond : V(G) \to \{c_a, c_b\}$ of $G^\diamond$ exists with $C^\diamond(v) = c_a$ on the even-parity layers and $C^\diamond(v) = c_b$ on the odd-parity layers (or vice versa). The agreement condition $C(v) = C^\diamond(v)$ on $C^{-1}(c_a) \cup C^{-1}(c_b)$ is then equivalent to the stated containment. □

Remark 2.4. The conjecture below asserts a structural property of 4-colorings of maximal planar graphs strictly stronger than the conclusion of the Four Color Theorem [1, 2]: it requires not merely the existence of a proper 4-coloring, but the existence of a proper 4-coloring together with a root u such that two of the four color classes are separated by the parity of the BFS layering from u.

Conjecture 2.5. *Every maximal planar graph G has a plane diamond coloring.*

References

[1] K. Appel and W. Haken, *Every planar map is four colorable*, Illinois Journal of Mathematics, vol. 21, no. 3, pp. 429–567, 1977.

[2] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, *The four-colour theorem*, Journal of Combinatorial Theory, Series B, vol. 70, no. 1, pp. 2–44, 1997.