

EVEN LEVEL GRAPH GENERATORS

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ABSTRACT.

1. INTRODUCTION

2. DEFINITIONS

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G . We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source). A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 2.2 (Levels). Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \text{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

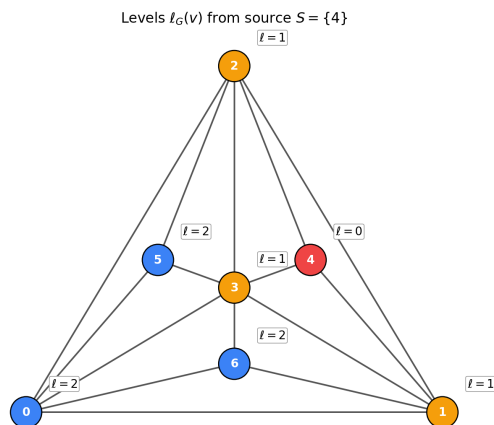


FIGURE 1. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

Definition 2.3 (Level cycle). A *level cycle* of G (with respect to a level source S) is a simple cycle in G all of whose vertices have the same level.

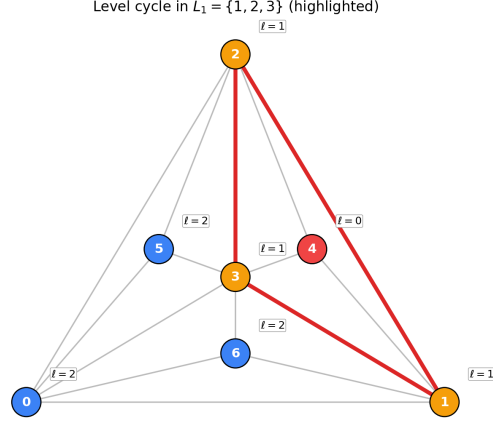


FIGURE 2. A level cycle in the triangulation of Figure 1. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

Definition 2.4 (Edge switch). Let G be a triangulation with level source S , and let $e = uv$ be an edge of a level cycle of G . The *edge switch* at e is the edge flip on e : writing uvw and uvx for the two triangular faces of G containing e , the edge uv is removed and the edge wx is added. As with any edge flip, the result is a triangulation on the same vertex set provided w and x are non-adjacent in G .

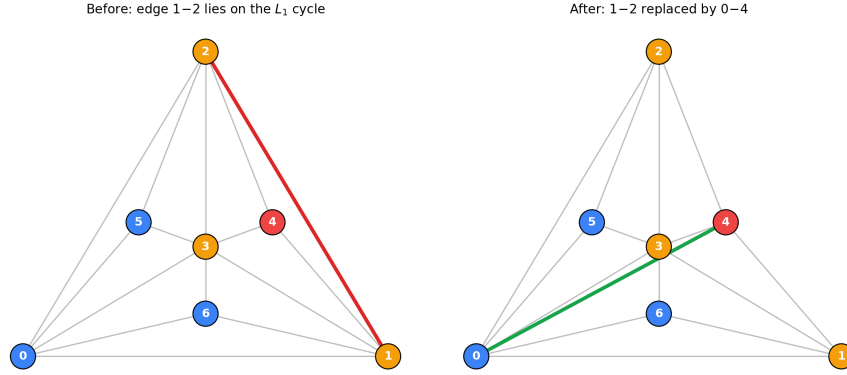


FIGURE 3. An edge switch on the level cycle of Figure 2. The chosen cycle edge 1–2 is shared by the triangular faces $(0, 1, 2)$ and $(1, 2, 4)$; the switch deletes 1–2 (red, left) and inserts 0–4 (green, right). Vertex colours indicate the original levels in G .

Definition 2.5 (Parity subgraph). Let G be a triangulation with level source S , and let G' be a triangulation on the same vertex set as G . The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod{2}\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G .

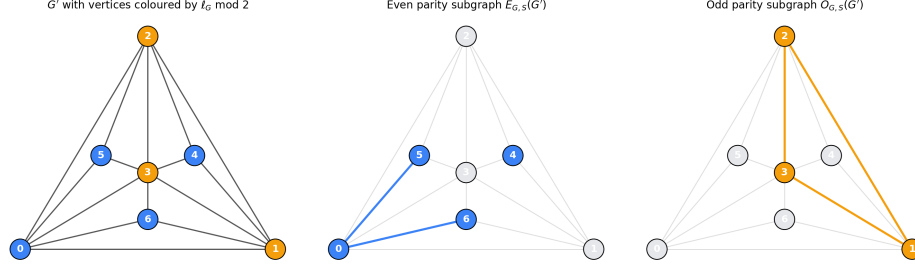


FIGURE 4. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 1 (here we take $G = G' = T$). Left: T with vertices coloured by $\ell_G \bmod 2$ (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0, 4, 5, 6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1, 2, 3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of G' .

3. OUTERPLANARITY OF LEVEL COMPONENTS

For each integer $k \geq 0$ and each (G, S) , write L_k for the subgraph of G induced by the level- k vertices. A *level component* of G (with respect to S) is a connected component of some L_k .

Theorem 3.1. *For every plane triangulation G and every level source S of G , every level component of G is outerplanar.*

Proof. Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph L_k is outerplanar. For $k = 0$, $L_0 = S$ is a single vertex and is trivially outerplanar.

Fix $k \geq 1$ and let D_k be the drawing of L_k inherited from Π_G . Let F^* be the face of D_k containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on ∂F^* , so u lies on the boundary of some other face of D_k . Take any path P in G from $v_0 \in S$ to u . As a curve in Π_G , P starts in F^* and ends at a point off ∂F^* , so it must transition from F^* to a different face of D_k ; in a planar embedding this can happen only at a vertex of D_k , that is, at a level- k vertex w on P . Either $w \neq u$ (so P has length $\geq \text{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every S -to- u path has length $\geq k + 1$, $\text{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. \square

4. EVEN LEVEL GRAPHS

Definition 4.1 (Even Level Graph). A plane triangulation G with level source S is an *Even Level Graph* if every level cycle of G has even length.

Theorem 4.2. *Every Even Level Graph is 4-colorable.*

Proof. Since adjacent vertices in G have levels differing by at most 1, any edge between two same-parity endpoints in fact connects two vertices at the same level. Hence

$$E_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i}, \quad O_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i+1},$$

and each L_k is bipartite because its cycles are level cycles of G , which have even length by hypothesis. Choose a 2-coloring of $E_{G,S}(G)$ in $\{\text{red}, \text{blue}\}$ and a 2-coloring of $O_{G,S}(G)$ in $\{\text{yellow}, \text{green}\}$. Same-parity edges of G are properly colored by the respective bipartition; opposite-parity edges connect $\{\text{red}, \text{blue}\}$ to $\{\text{yellow}, \text{green}\}$. The combined assignment is a proper 4-coloring of G . \square

Definition 4.3 (Derived level graph). Let G be an Even Level Graph with level source S , and let E and O denote the edge sets of the even and odd parity subgraphs $E_{G,S}(G)$ and $O_{G,S}(G)$. A *derived level graph* of G is a triangulation G' on the same vertex set as G obtained by a sequence of edge switches (Definition 2.4), each acting on an edge of E or of O . We do not update E or O to reflect the level structure of intermediate triangulations: throughout the sequence, an edge is classified as belonging to E (resp. O) if and only if both of its endpoints have even (resp. odd) level in G .

A derived level graph G' is *valid* if both $E_{G,S}(G')$ and $O_{G,S}(G')$ contain only even cycles.

Definition 4.4 (Bridge switch). Let G' be a triangulation reached from an Even Level Graph G , with parity classes inherited from G as in Definition 4.3. An edge switch on an edge $e \in E \cup O$ of G' , replacing uvw, uvx by the edge wx , is a *bridge switch* if either

- the new edge wx is a cross-parity edge (one endpoint even, the other odd), so wx enters neither parity subgraph; or
- wx is a same-parity edge and is a *bridge* in the parity subgraph it joins – that is, w and x lie in different connected components of that parity subgraph, so adding wx creates no new cycle.

Definition 4.5 (Bridge-derived level graph). A *bridge-derived level graph* of an Even Level Graph G is a triangulation obtained from G by a sequence of bridge switches (Definition 4.4).

Because a bridge switch never closes a cycle in a parity subgraph, it never introduces an odd cycle there. As an Even Level Graph has bipartite parity subgraphs (every level cycle is even), every bridge-derived level graph has bipartite parity subgraphs as well, and so is automatically a valid derived level graph. Equivalently, the first Betti number of each parity subgraph is non-increasing along any sequence of bridge switches.

Definition 4.6 (Intertwining tree). A maximal planar graph G is an *intertwining tree* if its vertex set can be partitioned into two sets A and B such that both induced subgraphs $G[A]$ and $G[B]$ are trees.

Theorem 4.7. *A maximal planar graph G is an intertwining tree if and only if its dual G^* has a Hamiltonian cycle.*

Proof. (\Rightarrow) Let $V(G) = A \sqcup B$ with $G[A]$ and $G[B]$ trees. Every triangular face $\{x, y, z\}$ of G meets both A and B : if all three vertices were in A the triangle would be a cycle in the tree $G[A]$, and likewise for B . Draw a closed curve through the faces of G separating the A -vertices from the B -vertices within each face. Since every face is split, the curve visits every face exactly once and crosses an edge of G precisely when that edge joins A to B ; it is therefore a Hamiltonian cycle of G^* .

(\Leftarrow) Let H be a Hamiltonian cycle of G^* . Drawn in the plane, H is a Jordan curve visiting every face of G once; let A and B be the vertices of G interior and exterior to H . The $2n - 4$ edges of H cross exactly the edges of G between A and B , leaving $(3n - 6) - (2n - 4) = n - 2$ edges inside $G[A]$ and $G[B]$ together. The edges inside A lie in the disk bounded by H and span A without enclosing a face (each face is cut by H), so $G[A]$ is a tree; likewise $G[B]$. \square

Conjecture 4.8. Every maximal planar graph is a valid derived level graph of some Even Level Graph, an intertwining tree, or both.

By Theorem 4.7, the intertwining-tree disjunct fails for G exactly when G^* is a counterexample to Tait’s conjecture. The smallest such G^* have 38 vertices (Holton–McKay [1], exactly 6 graphs), so the smallest triangulations that are not intertwining trees occur at $n = 21$ and there are exactly 6 of them. Below $n = 21$ every maximal planar graph is an intertwining tree, which is why the disjunction holds trivially in that range.

Empirical status. For each isomorphism class of maximal planar graphs on n vertices, we ask whether (i) some isomorphic representative is reachable from some Even Level Graph via E/O -edge switches (“derived”), and/or (ii) it is an intertwining tree. The conjecture holds for the class iff at least one of (i), (ii) holds.

| n | # iso | derived only | inter. only | both | missing | status |
|-----|-------|--------------|-------------|------|---------|--------|
| 6 | 2 | 0 | 0 | 2 | 0 | holds |
| 7 | 5 | 0 | 0 | 5 | 0 | holds |
| 8 | 14 | 0 | 0 | 14 | 0 | holds |
| 9 | 50 | 0 | 1 | 49 | 0 | holds |
| 10 | 233 | 0 | 0 | 233 | 0 | holds |
| 11 | 1249 | 0 | 0 | 1249 | 0 | holds |
| 12 | 7595 | 0 | 1 | 7594 | 0 | holds |

The boundary case $n = 21$. The first triangulations that are *not* intertwining trees are the six duals of the Holton–McKay graphs, at $n = 21$. For the disjunction to survive at $n = 21$, each of these six must be a valid derived level graph. We find:

- All six duals are confirmed not intertwining trees (exhaustive check of all $2^{20} - 1$ vertex bipartitions), consistent with Theorem 4.7.
- Two of the six are themselves Even Level Graphs (for a suitable source vertex), hence trivially valid derived level graphs. So the disjunction holds for them through the derived-level-graph disjunct – the first instances where that disjunct does work the intertwining-tree disjunct cannot.
- The remaining four are not Even Level Graphs for any source, and their full E/O -orbits ($\sim 10^8$ states per source labelling) are far too large to exhaust. Restricting to *bridge switches* (Definition 4.4) shrinks the relevant orbits by roughly two orders of magnitude and, crucially, keeps every reachable triangulation valid. A backward bridge-switch search over the valid parity partitions found an Even Level Graph witness for each of the four, so all four are *bridge-derived level graphs* (Definition 4.5) and hence valid derived level graphs. The witnessing orbits are small – between a few hundred and $\sim 1.7 \times 10^5$ states – even though other parity partitions of the same triangulations have orbits exceeding 10^6 ; finding one good partition suffices.

Thus at $n = 21$ the disjunction is confirmed for all six critical iso classes: two are Even Level Graphs outright, and the other four are bridge-derived level graphs. The bridge-switch restriction is what made the search tractable – it both shrinks the orbit and guarantees validity, so any Even Level Graph located in a backward orbit is an immediate witness.

REFERENCES

- [1] D. A. Holton and B. D. McKay. *The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices*. Journal of Combinatorial Theory, Series B, 45(3):305–319, 1988.