

# Cut tires form a tree (under depth nesting)

## The claim

**Proposition** (Cut tires form a forest). *For each side  $i$  of a 6-edge cut of  $G'$ , the cut tires of  $G'_i$ , parameterised by pairs  $(d, f)$  with  $d \geq 1$  and  $f$  a face of  $H_d$ , form a forest under the parent-child relation*

$$\text{parent}(T_{d+1}^{(f')}) := T_d^{(f)}$$

where  $f$  is the unique face of  $H_d$  in whose planar interior  $f'$  lies in the inherited embedding of  $G'_i$ .

The forest's roots are the cut tires at depth 1 (one per face of  $H_1$ ); their “virtual parent” is the cut  $C$  itself.

*Sketch.*  $H_{d+1}$  is a subgraph of  $G'_i$  with the inherited planar embedding. Each face of  $H_{d+1}$  is a maximal connected open region of  $|\Pi| \setminus E(H_{d+1})$  in the plane.

In particular, every face of  $H_{d+1}$  lies inside some face of  $H_d$  (since  $H_d$  has fewer edges and so larger faces). “Lies inside” means: the open face region of  $H_{d+1}$  is a subset of an open face region of  $H_d$ . This containment is unique because the faces of  $H_d$  partition  $|\Pi| \setminus E(H_d)$ .

Hence parent is well-defined and unique. No face of  $H_{d+1}$  is its own parent (because  $d+1 > d$ ). The relation defines a forest.

The roots are the depth-1 cut tires. Their “virtual parent” is the depth-0 pendant configuration, i.e. the cut  $C$  itself.  $\square$

## Why this matters for the chain half

Chain pigeonhole asks whether the per-tire  $S_3$ -orbit structure composes coherently through the chain. With a tree structure on the cut tires, this becomes a **tree dynamic-programming problem**, not a general graph compatibility problem:

- Process tires from leaves to root.
- At each leaf:  $\pi(T_{\text{leaf}})$  has known structure (e.g.  $S_3$ -orbits) from the per-tire half.
- Internal node  $T_d^{(f)}$  combines:
  - Its own internal  $\pi(T)$  structure.
  - Compatibility with each child  $T_{d+1}^{(f')}$  via the bijection  $\{\text{in spokes of } T_d^{(f)}\} \leftrightarrow \{\text{face boundary edges of } T_{d+1}^{(f')}\}$
- Root:  $T_1^{(\cdot)}$  projects its out-spoke colours to  $\sigma_i \in \mathcal{R}_i$ .

Tree DP is well-understood:  $|\mathcal{R}_i|$  can be computed exactly in linear time in the tree size (with size- $|\pi|$  tables at each node). Whether the resulting  $\mathcal{R}_0$  and  $\mathcal{R}_1$  intersect is a finite check at the cut.

The tree structure is also a **strong topological constraint** on the chain pigeonhole obstruction: any counterexample to chain pigeonhole at the cut must come from a tree-DP failure, which is much narrower than a general-graph obstruction.

## Empirical demonstration on Holton-McKay #0

$G'_1$  side ( $|S| = 28$ , depths 0 to 7)

Two depth-1 roots:

- Root (1,0): face length 12, no children (the outer “shell” of  $H_1$ ).
- Root (1,1): face length 4, with substantial subtree:
  - (2,0)  $|f| = 7$ 
    - \* (3,0)  $|f| = 2 \Rightarrow (4,0)$   $|f| = 4 \Rightarrow (5,0)$   $|f| = 14$
    - \* (3,1)  $|f| = 2 \Rightarrow (4,1)$   $|f| = 8 \Rightarrow (5,1)$   $|f| = 2 \Rightarrow (6,0)$   $|f| = 12 \Rightarrow (7,0)$   $|f| = 2$
    - \* (3,2)  $|f| = 2$
  - (2,1)  $|f| = 7$

$G'_0$  side ( $|S| = 10$ , depths 0 to 2)

Two depth-1 roots:

- Root (1,0): face length 9, with one child (2,0) ( $|f| = 6$ ).
- Root (1,1): face length 9, no children.

## Caveats on the empirical parent identification

The empirical demonstration used a vertex-sharing heuristic to identify parents: a face  $f'$  of  $H_{d+1}$  shares vertices with a face  $f$  of  $H_d$ , and we picked the parent as the one with smallest face length. This gives ambiguous candidates in some cases (8 ambiguous cases observed in  $G'_1$ ) because vertex sharing does not fully determine geometric containment.

A rigorous parent test would use *point-in-region* containment: pick a point in the open face of  $H_{d+1}$  (e.g., the centroid of its boundary walk), determine which face of  $H_d$  that point lies in (via the planar embedding’s face structure). This always gives a unique answer.

The ambiguity in our empirical run doesn’t reflect a violation of the proposition — it’s an artifact of the heuristic. Despite the ambiguity, the resulting tree structure looked sensible in both  $G'_0$  and  $G'_1$ .

## Consequence: the chain half becomes tractable

With the tree structure established (or assumed), the chain half of the loose chain pigeonhole conjecture reduces to:

**Reformulated chain half (tree DP form).** For each leaf cut tire  $T_{\text{leaf}}$ ,  $\pi(T_{\text{leaf}})$  is non-empty and  $S_3$ -closed. Propagating bottom-up through the parent-child relation preserves  $S_3$ -closure and non-emptiness. At the root depth-1 tires,  $\mathcal{R}_i$  is the join of the root tires' out-spoke projections. If  $\mathcal{R}_i$  is  $S_3$ -closed and contains a full  $S_3$ -orbit on each side, then  $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$  (containing a common orbit by  $S_3$ -equivariance).

The remaining questions:

1. Is non-emptiness preserved through parent-child propagation?
2. Is  $S_3$ -closure preserved? (Yes, by  $S_3$ -equivariance of the proper edge 3-colouring constraint.)
3. Does the join of root projections contain a full  $S_3$ -orbit?

Each of these is now a finite tree DP claim, much more tractable than the original “compose through the chain” formulation.

## Next step

1. Prove Proposition rigorously using the point-in-region containment definition of parent.
2. Implement the tree DP empirically on the Holton-McKay graphs and confirm  $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$  at the cut.
3. Attempt an analytical bound:  $|\mathcal{R}_i| \geq \text{somefunctionoftreesize}$ , ensuring  $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$  in general.