

# COLORING NESTED TIRE GRAPHS

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ABSTRACT.

## 1. INTRODUCTION

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation  $G$  is properly 4-vertex-colourable if and only if its dual cubic graph  $G'$  is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

We study the structure such a minimal counterexample would have to exhibit through the lens of *nested level duals*. Fixing a level source  $S$  in  $G$  endows the dual  $G'$  with a Breadth-First-Search-derived labelling, the dual depth of Definition 1.4, and the level structure of  $G$  organises  $G'$  into a family of nested cycles carrying these labels. Our aim is to express the obstruction to a 3-edge-colouring of  $G'$  as conditions on this nested labelled-cycle structure.

Throughout,  $G = (V, E)$  is a plane maximal planar graph (a triangulation) with a fixed planar embedding  $\Pi_G$ . We write  $|V| = n$ , so  $|E| = 3n - 6$  and  $G$  has  $2n - 4$  triangular faces.

**Definition 1.1** (Level source). A *level source* of  $G$  is any vertex  $v \in V$ ; we write  $S = \{v\}$  for the level-0 source.

**Definition 1.2** (Levels). Given a level source  $S \subseteq V$ , the *level* of  $v \in V$  is  $\ell_G(v) = \text{dist}_G(v, S)$ , the graph distance from  $v$  to the nearest source vertex.

**Definition 1.3** (Dual). The *dual* of  $G$ , written  $G'$ , is the inner (weak) planar dual of  $G$  with respect to the embedding  $\Pi_G$ : it has one vertex  $d_f$  for each bounded face  $f$  of  $G$ , and an edge joining  $d_f$  and  $d_{f'}$  for each edge of  $G$  shared by two bounded faces  $f$  and  $f'$ . The unbounded outer face contributes no vertex, and edges of  $G$  on the outer boundary contribute no dual edge. Since  $G$  is a triangulation, each vertex  $d_f \in V(G')$  corresponds to a triangular face  $f$  of  $G$ , and we write  $V(f) \subseteq V$  for its three incident vertices.

**Definition 1.4** (Dual depth). Given a level source  $S \subseteq V$ , the *dual depth* of a dual vertex  $d_f \in V(G')$  is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \text{dist}_G(v, S),$$

the smallest level among the three vertices of  $G$  bounding the face  $f$ .

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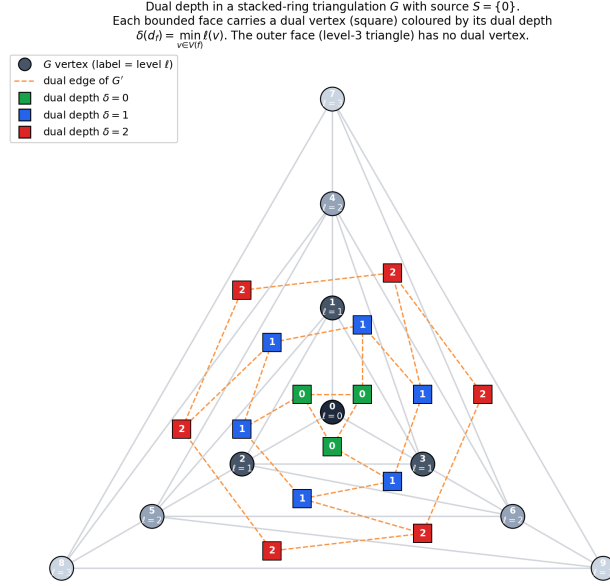


FIGURE 1. Dual depth in a stacked-ring triangulation  $G$  with level source  $S = \{0\}$ . Each  $G$  vertex is labelled by its level  $\ell$ . Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth  $\delta(d_f) = \min_{v \in V(f)} \ell(v)$ : the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

**Definition 1.5** (Tire graph). A *tire graph* consists of a plane graph  $T$  together with an *outer boundary*  $B_{\text{out}} \subseteq T$  and an *inner outerplanar graph*  $O \subseteq T$  with  $V(B_{\text{out}}) \cap V(O) = \emptyset$ , where

- $B_{\text{out}}$  is either a simple cycle of length  $\geq 3$  or a single vertex (a *degenerate outer boundary*);
- $O$  is an outerplanar graph; its *inner boundary*  $B_{\text{in}}$  is the closed walk in  $O$  that traces the boundary of  $O$ 's outer face in the inherited embedding, which is a simple cycle when  $O$  is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if  $|V(O)| = 1$ , we say  $T$  has a *degenerate inner boundary*.

At most one of  $B_{\text{out}}, B_{\text{in}}$  may be degenerate. The vertex and edge sets of  $T$  are

$$V(T) = V(B_{\text{out}}) \cup V(O), \quad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where  $E_{\text{ann}}$  — the *annular edges* — has the property that, in the plane embedding of  $T$ , the closed planar region  $R$  bounded externally by  $B_{\text{out}}$  and internally by  $B_{\text{in}}$  is partitioned into triangular faces of  $T$  whose union is  $R$ .

When  $B_{\text{out}}$  is a simple cycle and  $O$  is 2-connected,  $R$  is a closed annulus. More generally,  $R$  is a closed planar region that may fail to be a 2-manifold at cut-vertices of  $O$  (where two “lobes” of the depth- $d$  region meet at a single vertex); the inner boundary  $B_{\text{in}}$  is then a non-simple closed walk that visits the cut-vertex

multiple times. The relaxed definition accommodates outerplanar inner graphs with bridges, cut-vertices, or multiple connected components. When either boundary is degenerate,  $R$  is a closed disk with that vertex as apex.

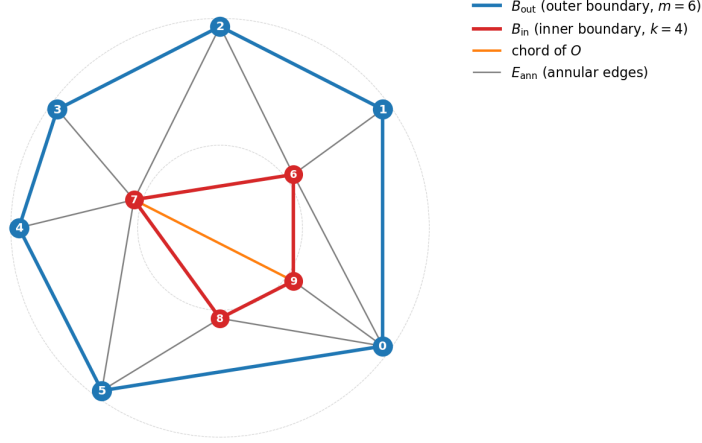


FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary  $B_{\text{out}}$  a 6-cycle on vertices  $0, \dots, 5$  (blue), inner boundary  $B_{\text{in}}$  a 4-cycle on vertices  $6, \dots, 9$  (red), inner outerplanar graph  $O = B_{\text{in}} \cup \{7-9\}$  (with one chord, orange), and  $E_{\text{ann}}$  (grey) tiling the annulus between  $B_{\text{out}}$  and  $B_{\text{in}}$  by ten triangular faces.

*Remark 1.6.* Let  $m = |V(B_{\text{out}})|$  and  $k = |V(B_{\text{in}})|$ . By Euler's formula on the annular (resp. disk) region  $R$ , the tire graph has  $m+k$  triangular faces inside  $R$  and  $|E_{\text{ann}}| = m+k$  annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so  $\min(m, k) = 1$ ), there are  $m+k-1$  triangular faces and  $|E_{\text{ann}}| = m+k-1$ .

**Definition 1.7** (Partial tire dual). Let  $T = (B_{\text{out}}, O, E_{\text{ann}})$  be a tire graph in the sense of Definition 1.5, and let  $F_{\text{ann}}$  denote the set of triangular faces of  $T$  in the closed annular region between  $B_{\text{out}}$  and  $B_{\text{in}}$ . The *partial tire dual* of  $T$ , written  $D(T)$ , is the graph defined as follows.

*Vertices.*

- (V1) For each face  $f \in F_{\text{ann}}$ , an *interior vertex*  $d_f$  of  $D(T)$ .
- (V2) For each edge  $e \in E(B_{\text{out}})$ , a *leaf vertex*  $\ell_e^{\text{out}}$ .
- (V3) For each occurrence of an edge in the closed walk  $B_{\text{in}}$  (= the outer-face boundary walk of  $O$ ), a *leaf vertex*  $\ell_e^{\text{in}}$ . (When  $O$  is 2-connected each edge appears once; cut-vertices and bridges of  $O$  may cause an edge or vertex to appear more than once.)

*Edges.*

- (E1) For each edge  $e \in E(T)$  whose two incident faces both lie in  $F_{\text{ann}}$  (an *interior annular edge*), one edge  $\{d_{f_1}, d_{f_2}\} \in E(D(T))$  where  $f_1, f_2 \in F_{\text{ann}}$  are the two annular faces incident to  $e$ .

- (E2) For each  $e \in E(B_{\text{out}})$ , one edge  $\{d_f, \ell_e^{\text{out}}\} \in E(D(T))$  where  $f \in F_{\text{ann}}$  is the unique annular face incident to  $e$ . The leaf  $\ell_e^{\text{out}}$  has degree 1.
- (E3) For each occurrence of  $e$  on the boundary walk  $B_{\text{in}}$ , one edge  $\{d_f, \ell_e^{\text{in}}\} \in E(D(T))$  where  $f \in F_{\text{ann}}$  is the annular face incident to  $e$  on the side of that occurrence. The leaf  $\ell_e^{\text{in}}$  has degree 1.

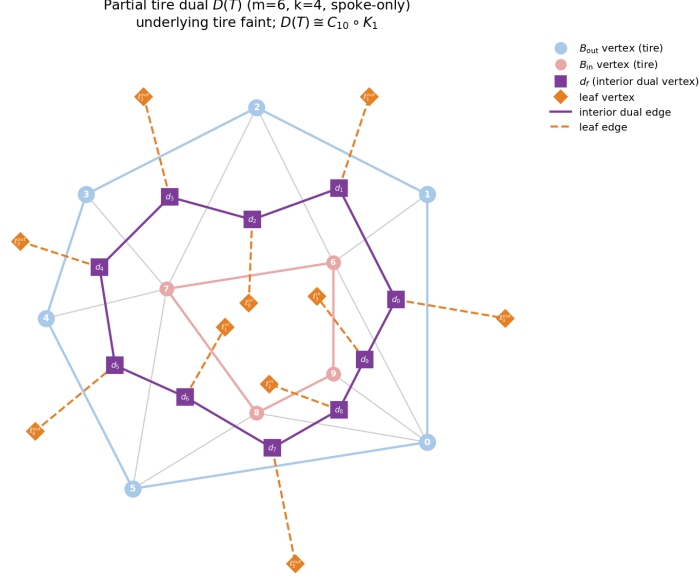


FIGURE 3. The partial tire dual  $D(T)$  (purple squares + orange diamonds) drawn on top of a small tire graph  $T$  (faint) with  $m = 6$  and  $k = 4$ . The ten interior vertices  $d_f$  at the centroids of the annular triangles form a single 10-cycle (solid purple); each boundary edge of the annular region (either of  $B_{\text{out}}$  or of  $B_{\text{in}}$ ) contributes a degree-1 leaf (orange diamond) attached to the unique annular face incident to it (dashed orange), giving the structure  $C_{10} \circ K_1$  of Proposition 1.8.

**Proposition 1.8** (Structure of  $D(T)$  when the annular triangulation is spoke-only). *Suppose  $B_{\text{out}}$  is a simple cycle of length  $n$ ,  $O$  is a 2-connected outerplanar graph whose outer-face cycle  $B_{\text{in}}$  has length  $m$ , and  $E_{\text{ann}}$  consists only of spokes (edges with one endpoint in  $V(B_{\text{out}})$  and one in  $V(B_{\text{in}})$ ). Then each face  $f \in F_{\text{ann}}$  has exactly one boundary edge (on  $B_{\text{out}}$  or  $B_{\text{in}}$ ) and two interior annular edges, and consequently  $D(T)$  is isomorphic to the corona graph  $C_{n+m} \circ K_1$ : a cycle of length  $n + m$  on the interior vertices  $\{d_f\}$ , with one leaf attached to each cycle vertex.*

*In particular,  $|V(D(T))| = 2(n + m)$  and  $|E(D(T))| = 2(n + m)$ .*

*Proof.* Each annular triangle  $f$  in a spoke-only triangulation has the form  $\{x, y, z\}$  with  $x \in V(B_{\text{out}})$ ,  $y \in V(B_{\text{in}})$ , and  $z$  also in  $V(B_{\text{out}}) \cup V(B_{\text{in}})$ . Of its three edges, the one between the two same-side vertices ( $x$ - $z$  if both on  $B_{\text{out}}$ , or  $y$ - $z$  if both on  $B_{\text{in}}$ ) is a boundary edge of the annular region; the other two edges are spokes.

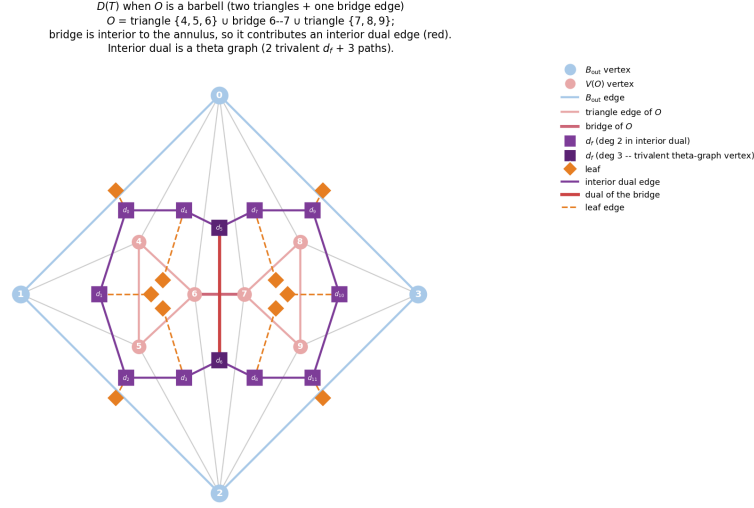


FIGURE 4. Partial tire dual  $D(T)$  when the inner outerplanar graph  $O$  has a bridge — here a non-trivial edge cut connecting two disjoint triangles.  $B_{\text{out}}$  is a 4-cycle on  $\{0, 1, 2, 3\}$  and  $O$  is the barbell: triangle  $\{4, 5, 6\}$  together with triangle  $\{7, 8, 9\}$  joined by the bridge edge 6–7 (removing the bridge disconnects  $O$ ). Because both faces incident to the bridge are annular triangles, the bridge contributes an *interior dual edge* (highlighted in red) rather than two leaves; consequently the interior dual subgraph is no longer the single  $(n + m)$ -cycle of Proposition 1.8, but a theta graph: the two trivalent vertices  $d_5, d_6$  (the bridge-incident annular faces) are joined by three internally vertex-disjoint paths in  $D(T)$ . Leaves come only from  $B_{\text{out}}$  ( $n = 4$  leaves) and the six non-bridge edges of  $O$  ( $m_{\partial} = 6$  leaves, three for each triangle).

So each  $d_f$  has degree 3 in  $D(T)$ : two from interior edges (= spokes shared with adjacent annular faces) and one leaf. The induced subgraph on  $\{d_f : f \in F_{\text{ann}}\}$  is 2-regular; together with the connectedness of the annular region this forces it to be a single cycle. By Remark 1.6, the cycle has length  $n + m$ , and there are also  $n + m$  leaves attached one-per-cycle-vertex.  $\square$

**Proposition 1.9** (Source-side simple-cycle property). *Let  $G$  be a maximal planar graph with planar embedding  $\Pi_G$  and single-vertex source  $v_0$ . Let  $d \geq 1$ ,  $v \in L_d$ , and let  $C'$  be a connected component of  $G'_d$  such that  $v$  is incident to some face in  $F_{C'}$ . Then the depth- $d$  faces in  $F_{C'}$  incident to  $v$  form a single contiguous arc in  $v$ 's rotation in  $\Pi_G$ .*

*Equivalently: for any such component, the source-side boundary of  $R_{C'}$  is a simple cycle in  $L_d$  (no cut-vertices at level  $d$ ).*

*Proof.* Suppose for contradiction that the depth- $d$  faces in  $F_{C'}$  at  $v$  lie in two or more disjoint arcs of  $v$ 's rotation. Adjacent vertices in  $G$  differ in level by at most 1, so a face at  $v$  has depth exactly  $d$  iff both other vertices have level  $\geq d$ , and

depth  $\leq d - 1$  iff at least one has level  $d - 1$ . Hence the gaps between the depth- $d$  arcs at  $v$  are populated by level- $(d - 1)$  neighbours of  $v$ , occurring in at least two disjoint arcs of  $v$ 's rotation. Pick  $p$  in one such gap and  $q$  in another.

The BFS ball  $G[L_{<d}]$  is connected, so there exists a simple path  $P$  in  $G[L_{<d}]$  from  $p$  to  $q$ . Define the closed walk

$$W := v \rightarrow p \rightarrow P \rightarrow q \rightarrow v.$$

Every vertex of  $P$  lies in  $L_{<d}$ , while  $\ell(v) = d$ , so  $v$  is distinct from every vertex of  $P$ ;  $P$  is simple, so its internal vertices are distinct; and  $p \neq q$  since they lie in different gaps. Hence  $W$  is a simple cycle in  $G$ .

By the Jordan curve theorem, the planar embedding of  $W$  divides  $\Pi_G$  into two regions. In  $v$ 's rotation, the edges  $v - p$  and  $v - q$  lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by  $W$ . By choice of  $p, q$ , the two arcs of depth- $d$  faces at  $v$  in  $F_{C'}$  lie in different regions of  $W$  (i.e., one arc on each side).

Since  $C'$  is connected in  $G'$  and contains depth- $d$  faces in both arcs, there is a dual path  $f_1, f_2, \dots, f_k$  in  $G'_d$  with  $f_1, f_k \in F_{C'}$  incident to  $v$  in different arcs, and with the intermediate faces  $f_2, \dots, f_{k-1}$  not incident to  $v$  (a shortest such dual path). Consecutive faces  $f_i, f_{i+1}$  share an edge  $e_i$  of  $G$ ; for  $i \geq 2$ , both endpoints of  $e_i$  lie in  $L_{\geq d}$  (since neither  $f_i$  nor  $f_{i+1}$  is incident to  $v$ , all six vertices of these two triangles lie in  $L_{\geq d}$ ). In particular,  $e_i$  shares no endpoint with  $W$  except possibly  $v$  — and  $v$  is excluded from  $f_2, \dots, f_{k-1}$ .

A planar edge with neither endpoint on a simple closed planar curve  $W$  has both of its incident faces on the same side of  $W$ . Applying this to each  $e_i$  ( $i \geq 2$ ) inductively: starting from  $f_2$  on the same side of  $W$  as  $f_1$  (their shared edge  $e_1 = w - w'$  opposite to  $v$  in  $f_1$  has  $w, w' \in L_{\geq d}$  and hence is not on  $W$ ), the path  $f_2 \rightarrow f_3 \rightarrow \dots \rightarrow f_{k-1} \rightarrow f_k$  stays on one side of  $W$ .

But  $f_1$  and  $f_k$  lie on different sides of  $W$  (by construction), contradicting the conclusion that the entire path lies on one side.  $\square$

**Lemma 1.10** (Tire-component lemma). *Let  $G$  be a maximal planar graph and let  $S \subseteq V(G)$  be a level source. Fix a plane embedding  $\Pi_G$  of  $G$  in which  $S$  lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For  $d \geq 0$ , let*

$$G'_d := G'[\{d_f \in V(G') : \delta_G(d_f) = d\}]$$

*be the inner-dual subgraph on dual vertices of dual depth  $d$ , and let  $C'$  be a connected component of  $G'_d$ . Write  $F_{C'} := \{f : d_f \in V(C')\}$ ,  $V_{C'} := \bigcup_{f \in F_{C'}} V(f)$ , and let  $C := G[V_{C'}]$  inherit its embedding from  $\Pi_G$ . Set  $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$ .*

*Then  $C$ , with the inherited embedding, is a tire graph in the sense of Definition 1.5. Its outer boundary  $B_{\text{out}}$  is the side of  $R_{C'}$  closer to  $S$  in  $\Pi_G$ , namely the level- $d$  subgraph  $G[V_{C'} \cap L_d]$  (a simple cycle or single vertex); its inner outerplanar graph is  $O = G[V_{C'} \cap L_{d+1}]$ , and its inner boundary  $B_{\text{in}}$  is the outer-face boundary closed walk of  $O$  in the inherited embedding (a simple cycle when  $O$  is 2-connected, a non-simple closed walk in general). The triangular faces of  $C$  inside the closed boundary region are exactly the faces of  $G$  in  $F_{C'}$ .*

*Proof. Outerplanarity of the two level parts.* By construction  $S$  lies on the outer face of  $\Pi_G$ , so the outerplanarity lemma of [1] applies directly with  $(G, \Pi_G, S)$ ,

giving that  $G[L_{d'}]$  is outerplanar for each  $d' \geq 0$ . Subgraphs of outerplanar graphs are outerplanar, so  $G[V_{C'} \cap L_d]$  and  $G[V_{C'} \cap L_{d+1}]$  are both outerplanar.

*Layer containment.* Each  $f \in F_{C'}$  has at least one vertex at level  $d$ , and adjacent vertices in  $G$  differ in level by at most 1; combined with  $\delta_G(d_f) = d$ , this forces  $V(f) \subseteq L_d \cup L_{d+1}$ . Hence  $V_{C'} \subseteq L_d \cup L_{d+1}$ , and  $C$  has vertex partition  $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$ .

*Boundary edges are monochromatic in level.* Each edge  $e$  on  $\partial R_{C'}$  separates a face  $f \in F_{C'}$  from a face  $f' \notin F_{C'}$ . Because  $f$  and  $f'$  share the edge  $e$ , their dual vertices are adjacent in  $G'$ ; if both had depth  $d$  they would lie in the same component of  $G'_d$ , contradicting  $d_f \in C'$  and  $d_{f'} \notin C'$ . Hence  $\delta_G(d_{f'}) \neq d$ ; combined with the bounded-step property of  $\delta$  across  $G'$ -adjacent faces,  $\delta_G(d_{f'}) \in \{d-1, d+1\}$ .

- If  $\delta_G(d_{f'}) = d-1$ , the third vertex  $w$  of  $f' = \{u, v, w\}$  (where  $u, v$  are the endpoints of  $e$ ) has  $\ell(w) = d-1$ . Each of  $u, v$  has  $\ell \in \{d, d+1\}$  (from  $V(f) \subseteq L_d \cup L_{d+1}$ ) and is adjacent to  $w$ , forcing  $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$ .
- If  $\delta_G(d_{f'}) = d+1$ , then all three vertices of  $f'$  lie in  $L_{\geq d+1}$ , so in particular  $\ell(u) = \ell(v) = d+1$ .

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of  $\partial R_{C'}$  is monochromatic in level.

*Boundary structure.* Each connected component of  $\partial R_{C'}$  traces a closed walk in  $G$  that, by the monochromaticity above, lies entirely in  $L_d$  or entirely in  $L_{d+1}$ . By Proposition 1.9, the depth- $d$  faces of  $F_{C'}$  at any  $v \in L_d \cap V_{C'}$  form a single contiguous arc in  $v$ 's rotation, so the source-side boundary walk visits each  $L_d$ -vertex of  $V_{C'}$  exactly once: it is a simple cycle. At vertices  $v \in L_{d+1} \cap V_{C'}$  the depth- $d$  faces may split into multiple arcs of  $v$ 's rotation; this corresponds exactly to  $v$  being a cut-vertex of  $O$ , and the inner-side boundary walk visits  $v$  correspondingly many times — which is already accommodated by Definition 1.5 (where  $B_{\text{in}}$  is the outer-face boundary closed walk of  $O$ , not necessarily a simple cycle).

*Outer boundary.* Because  $S$  lies on the outer face of  $\Pi_G$ , the boundary curve(s) of  $R_{C'}$  on the  $L_d$  side are closer to  $S$  in the embedding. In the inherited embedding of  $C$ , the unique unbounded face is the merged region containing the rest of  $\Pi_G$  outside  $R_{C'}$  on the  $S$  side, so its boundary — a simple cycle on  $L_d$  (or a single vertex when  $V_{C'} \cap L_d = \{v_0\}$ , the  $d = 0$  case) — serves as  $B_{\text{out}}$ . We set  $B_{\text{out}} := G[V_{C'} \cap L_d]$  if this is a cycle, and the single vertex  $\{v_0\}$  in the degenerate case.

*Inner outerplanar graph.* By the outerplanarity lemma of [1],  $G[V_{C'} \cap L_{d+1}]$  is outerplanar. We set  $O := G[V_{C'} \cap L_{d+1}]$ . The boundary curve(s) of  $R_{C'}$  on the  $L_{d+1}$  side are exactly the boundary of  $O$ 's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around  $O$  from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary  $B_{\text{in}}$ . No further restriction on  $O$ 's internal structure is needed: when  $R_{C'}$  has more than two boundary components in the surface-classification sense (i.e. several disjoint simple cycles on  $L_{d+1}$ ), these correspond precisely to the multiple connected components or bridge crossings of  $O$ , and the outer-face boundary closed walk of  $O$  captures them collectively.

*Tire structure.* The triangular faces of  $C$  inside the closed boundary region are by construction the depth- $d$  faces in  $F_{C'}$ , and the edges of  $C$  are  $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$

where  $E_{\text{ann}}$  are the edges of  $G$  between  $V_{C'} \cap L_d$  and  $V_{C'} \cap L_{d+1}$  that bound a face of  $F_{C'}$ .  $\square$

*Remark 1.11.* Either boundary part of  $C$  in Lemma 1.10 may be degenerate. At  $d = 0$  with single-vertex source  $S = \{v_0\}$  the unique component of  $G'_0$  has  $V_{C'} \cap L_0 = \{v_0\}$  as the degenerate *outer* boundary and  $V_{C'} \cap L_1$  a cycle (the link of  $v_0$  in  $G$ ) as the inner boundary. Symmetrically, at  $d = D_{\text{max}}$ ,  $V_{C'} \cap L_{D_{\text{max}}+1} = \emptyset$  degenerates to a single deepest vertex serving as the *inner* boundary, with the level- $D_{\text{max}}$  cycle as the outer boundary.

*Remark 1.12.* Two structural features of  $R_{C'}$  that might at first appear to obstruct the tire-graph conclusion are both already accommodated by Definition 1.5:

*Cut-vertices of  $O$ .* A vertex  $v \in V_{C'} \cap L_{d+1}$  may have the faces of  $F_{C'}$  incident to it split into two or more arcs in  $v$ 's rotation in  $\Pi_G$ , separated by faces of higher depth. This corresponds exactly to  $v$  being a cut-vertex of  $O = G[V_{C'} \cap L_{d+1}]$ , and the inner boundary closed walk  $B_{\text{in}}$  then visits  $v$  multiple times — once for each arc. No additional hypothesis is needed.

*Multi-hole topology of  $R_{C'}$ .* Even when  $R_{C'}$  encloses several disjoint depth- $> d$  sub-regions, the inner outerplanar graph  $O$  captures the multi-hole structure as a disconnected or non-2-connected outerplanar graph (with bridges or multiple components), and its outer-face boundary closed walk serves as  $B_{\text{in}}$  traversing bridges twice and visiting cut-vertices multiple times.

In the special case  $d = 0$  with single-vertex source  $S = \{v_0\}$ ,  $R_{C'}$  is the star of  $v_0$ , a topological closed disk with one boundary cycle (the link of  $v_0$ ); the corresponding tire graph has degenerate outer boundary  $\{v_0\}$ .

**Proposition 1.13** (Edge-vertex coloring bijection for  $D(T)$ ). *Let  $T$  be a tire graph satisfying the spoke-only hypothesis of Proposition 1.8 (so  $D(T) \cong C_{n+m} \circ K_1$ ). Let  $\Gamma \cong C_{n+m}$  be the interior dual subgraph of  $D(T)$  induced on the interior dual vertices  $\{d_f : f \in F_{\text{ann}}\}$ . Then the number of proper 3-edge-colorings of  $D(T)$  equals the number of proper 3-vertex-colorings of  $\Gamma$ , both given by*

$$2^{n+m} + 2 \cdot (-1)^{n+m}.$$

*Proof.* Write  $L = n + m$ ,  $\Gamma = C_L$ . We construct mutually inverse bijections.

*Step 1: proper 3-edge-colorings of  $D(T) \leftrightarrow$  proper 3-edge-colorings of  $C_L$ .* Given a proper 3-edge-coloring  $\chi$  of  $D(T)$ , the three edges incident to any  $d_f$  carry three distinct colors; in particular the two cycle edges incident to  $d_f$  carry distinct colors, so  $\chi|_{E(C_L)}$  is a proper 3-edge-coloring of  $C_L$ . Conversely, given a proper 3-edge-coloring  $\psi$  of  $C_L$ , the two cycle edges at any  $d_f$  have distinct colors, so a unique third color is available; assign that color to  $d_f$ 's leaf edge. The resulting extension to  $D(T)$  is proper at every  $d_f$  and vacuously proper at every leaf (degree 1), and the two maps are inverse to each other. Therefore

$$\#\{\text{proper 3-edge-colorings of } D(T)\} = \#\{\text{proper 3-edge-colorings of } C_L\}.$$

*Step 2: proper 3-edge-colorings of  $C_L \leftrightarrow$  proper 3-vertex-colorings of  $L(C_L) \cong C_L$ .* The line graph  $L(C_L)$  of a cycle of length  $L$  is again a cycle of length  $L$ ; proper edge-colorings of  $C_L$  are by definition proper vertex-colorings of  $L(C_L)$ .

*Step 3: count.* The chromatic polynomial of the cycle is  $P(C_L, k) = (k-1)^L + (-1)^L(k-1)$ ; at  $k = 3$  this gives  $2^L + 2 \cdot (-1)^L$ .  $\square$



*Remark 1.14.* Proposition 1.13 reduces counting proper 3-edge-colorings of  $D(T)$  to counting proper 3-vertex-colorings of a single cycle, giving a closed form  $2^{n+m} + 2(-1)^{n+m}$  that depends only on  $n + m$  (not on the specific spoke-only annular triangulation, nor on the chord structure of  $O$ ). The count is preserved under the corona-with- $K_1$  structure of Proposition 1.8 precisely because each degree-1 leaf imposes no proper-edge-coloring constraint on itself; its color is freely determined as the missing third color at its attached interior vertex.

**Definition 1.15** (Tire annular subgraph). Let  $G$  be a maximal planar graph with embedding  $\Pi_G$  and inner planar dual  $G'$  (as in Definition ?? above). Let  $T = (B_{\text{out}}, O, E_{\text{ann}}) \subseteq G$  be a tire graph (Definition 1.5), and let  $F_{\text{ann}} \subseteq F(G)$  denote its set of annular faces. The *tire annular subgraph* of  $T$  in  $G'$  is

$$T'_{\text{ann}} := G'[\{d_f : f \in F_{\text{ann}}\}],$$

the subgraph of  $G'$  induced on the dual vertices corresponding to the annular faces of  $T$ . We equip  $T'_{\text{ann}}$  with the planar embedding inherited from  $G'$  (which, by deletion of vertices outside the annulus, remains a planar embedding of  $T'_{\text{ann}}$  in the sense of  $\Pi_G$ ).

**Definition 1.16** (Tire annular face connector). With  $G, G', T$  as in Definition 1.15, let  $f'$  be a face of the tire annular subgraph  $T'_{\text{ann}}$  in its inherited embedding, and let  $V(f') \subseteq V(T'_{\text{ann}})$  denote the set of vertices on the boundary walk of  $f'$ . The *tire annular face connector at  $f'$*  is the subgraph

$$T'_{f'} := (V(f') \cup N_{G'}(V(f')), \{e \in E(G') : e \text{ is incident to } V(f')\}) \subseteq G',$$

i.e. the subgraph of  $G'$  on the closed  $G'$ -neighborhood of  $V(f')$  together with every  $G'$ -edge incident to  $V(f')$ .

**Definition 1.17** (Inner and outer spokes). With  $T'_{f'}$  as in Definition 1.16, regard  $f'$  as an open region of  $|\Pi_G|$  and write  $\overline{f'}$  for its closure. The vertices of  $V(T'_{f'}) \setminus V(f')$  lie in  $|\Pi_G| \setminus \overline{f'}$  or in  $f'$  (never on  $\partial f'$ , since the boundary walk of  $f'$  is by definition the set  $V(f')$ ). Partition

$$V(T'_{f'}) \setminus V(f') = V_{\text{out}}(T'_{f'}) \sqcup V_{\text{in}}(T'_{f'})$$

where

$$\begin{aligned} V_{\text{out}}(T'_{f'}) &:= \{v \in V(T'_{f'}) \setminus V(f') : v \notin \overline{f'}\}, \\ V_{\text{in}}(T'_{f'}) &:= \{v \in V(T'_{f'}) \setminus V(f') : v \in f'\}. \end{aligned}$$

The elements of  $V_{\text{out}}(T'_{f'})$  are the *outer spokes* of  $T'_{f'}$  (vertices not in  $V(f')$  that lie outside the region bounded by  $f'$ ); the elements of  $V_{\text{in}}(T'_{f'})$  are the *inner spokes* of  $T'_{f'}$  (vertices not in  $V(f')$  that lie inside the region bounded by  $f'$ ).

*Remark 1.18.* In the spoke-only setting of Proposition 1.8, the tire annular subgraph is  $T'_{\text{ann}} = \Gamma \cong C_{n+m}$  (Proposition 1.13). This cycle has exactly two faces in its inherited embedding – one on each side of the cycle in  $\Pi_G$  – and both face boundaries traverse all  $n + m$  vertices, so  $V(f') = V(\Gamma)$  for either choice of  $f'$ . Each interior dual vertex  $d_f$  has  $G'$ -degree 3 (since  $G$  is a triangulation), of which two edges lie in  $\Gamma$  (cycle edges) and one edge points to a single non-annular face of  $G$ . Consequently  $T'_{f'}$  has  $n + m$  interior vertices plus the non-annular face vertices to which they connect, and is independent of the choice of  $f'$ . When  $G$  consists

only of the tire  $T$  together with one source-side face inside  $B_{\text{out}}$  and one  $O$ -side face inside  $B_{\text{in}}$ ,  $T'_{f'}$  recovers the planar dual of  $T$  itself.

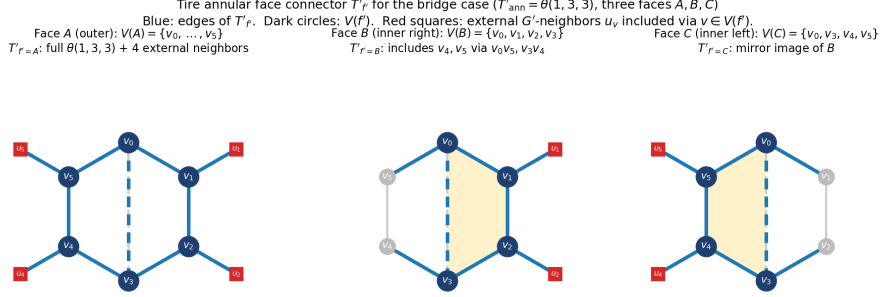


FIGURE 5. The bridge case:  $T'_{\text{ann}} = \theta(1, 3, 3)$  has three faces  $A, B, C$  in its inherited embedding, with respective vertex sets  $V(A) = \{v_0, \dots, v_5\}$ ,  $V(B) = \{v_0, v_1, v_2, v_3\}$ , and  $V(C) = \{v_0, v_3, v_4, v_5\}$ . In the surrounding maximal planar  $G$ , the chord endpoints  $v_0, v_3$  (the two annular faces sharing the bridge edge) have all three  $G'$ -edges inside  $T'_{\text{ann}}$ , while each non-chord vertex  $v_i$  ( $i \in \{1, 2, 4, 5\}$ ) contributes one  $G'$ -edge to an external non-annular neighbor  $u_i$ . Each panel highlights  $T'_{f'}$  (blue) inside  $G'$ : dark circles are  $V(f')$ , gray circles are  $G'$ -neighbors of  $V(f')$  within  $T'_{\text{ann}}$ , and red squares are external  $G'$ -neighbors  $u_i$ . The choice of face  $f'$  controls which external neighbors  $u_i$  are pulled into  $T'_{f'}$  (face  $A$  pulls in all four; face  $B$  pulls in  $u_1, u_2$  and face  $C$  pulls in  $u_4, u_5$ ).

## 2. A CONJECTURAL LATIN-STYLE SUBSTRUCTURE

Empirical enumeration (notes `tire_fiber_data.tex`, `tire_fiber_chords.tex`, `tire_fiber_step2.tex`, `tire_fiber_step2_large.tex`) of edge 3-coloring distributions on the tire annular face connector  $T'_{f'}$  across 46 adjacent-tire pairs at  $|\gamma| \in \{3, 4, 5, 6, 9, 12\}$  suggests that the chain-pigeonhole step on a shared cycle always succeeds. The data points to a structural mechanism: every edge-3-colourable tire admits at least one “Latin-flavoured” boundary configuration, and adjacent tires share this same substructure on their common cycle.

Concretely, fix a tire  $T$  with inner outerplanar graph  $O$  on  $V(B_{\text{in}})$  and let  $F(O)$  be the set of  $O$ -faces (in the tire’s plane embedding, not counting the outer face  $B_{\text{in}}$ ). For each  $O$ -face  $f \in F(O)$ , let  $E_{\text{in}}(f) \subseteq E(B_{\text{in}})$  denote the set of  $B_{\text{in}}$  edges on  $f$ ’s boundary. In the Steiner-poor surrounding triangulation (where each  $O$ -face is a single face of  $G$  and dualises to a single  $G'$ -vertex of degree  $|E_{\text{in}}(f)|$  in  $T'_{f'}$ ), proper edge 3-colouring of  $T'_{f'}$  requires every  $O$ -face to have  $|E_{\text{in}}(f)| \leq 3$ .

Let  $\sigma_{B_{\text{in}}}$  denote the spoke colouring restricted to the  $|V(B_{\text{in}})|$  inner-direction spoke positions on the dual annular cycle (equivalently:  $\sigma$  indexed by  $E(B_{\text{in}})$ ). Define the *Latin-flavoured set* on  $\gamma = B_{\text{in}}$  as

$$\mathcal{L}(B_{\text{in}}, O) := \left\{ \sigma : E(B_{\text{in}}) \rightarrow \{1, 2, 3\} \mid \sigma|_{E_{\text{in}}(f)} \text{ is a permutation of } \{1, 2, 3\} \text{ for every } f \in F(O) \right\}.$$

That is, on every  $O$ -face's  $B_{\text{in}}$ -edge boundary, all three colours appear exactly once (forcing  $|E_{\text{in}}(f)| = 3$  for each face — the maximally constrained case).

**Conjecture 2.1** (Latin-substructure conjecture). *For any Steiner-poor edge-3-colourable tire  $T$  with inner outerplanar graph  $O$  such that every  $O$ -face has exactly 3  $B_{\text{in}}$ -edges, the realisable inner-spoke projection  $\pi_D(\mathcal{C}(T'_{f'}))$  contains  $\mathcal{L}(B_{\text{in}}, O)$  as a subset. Moreover,  $\mathcal{L}(B_{\text{in}}, O)$  is invariant under the  $S_3$  action on colours and has size at least  $3! = 6$ .*

**Conjecture 2.2** (Chain-pigeonhole compatibility from Latin substructure). *Adjacent tires  $T_1, T_2$  sharing a cycle  $\gamma$  admit a joint edge 3-colouring whenever their respective inner-outerplanar structures  $O^{(1)}, O^{(2)}$  both satisfy Conjecture 2.1. Equivalently:  $\pi_D^{(1)}(\mathcal{C}(T_1)) \cap \pi_U^{(2)}(\mathcal{C}(T_2)) \supseteq \mathcal{L}(\gamma, O^{(1)}) \cap \mathcal{L}(\gamma, O^{(2)})$ , and this last intersection is non-empty whenever the two face partitions of  $E(\gamma)$  induced by  $O^{(1)}, O^{(2)}$  share a common “Latin completion.”*

The structural origin of these conjectures is the empirical observation that the smallest tested intersections on  $\gamma$  are always exactly the  $3!$  permutations of a single canonical pattern in which each  $O$ -face on  $\gamma$ 's side receives a permutation of  $\{1, 2, 3\}$ . For example, at  $|\gamma| = 12$  with  $O^{(1)}$  given by the chord matching  $\{(0, 3), (4, 7), (8, 11)\}$  (face structure  $\{0, 1, 2\} \sqcup \{4, 5, 6\} \sqcup \{8, 9, 10\} \sqcup \{3, 7, 11\}$ ), the canonical pattern  $(1, 2, 3, 2, 2, 1, 3, 3, 2, 3, 1, 1)$  assigns the permutation  $(1, 2, 3)$  to the first face,  $(2, 1, 3)$  to the second,  $(2, 3, 1)$  to the third, and  $(2, 3, 1)$  to the fourth. Every face receives all three colours.

A proof of Conjecture 2.1 would convert the chain-pigeonhole compatibility step into a structural theorem on  $T'_{f'}$ : it is not the rough abundance of valid spoke configurations that lets adjacent tires meet, but a specific Latin-square-flavoured substructure dictated by the face partition of each tire's inner outerplanar graph. See `notes/tire_fiber_step2_large.tex` for the data underlying this conjecture and `experiments/tire_fiber_counterexample_search.log` for the ongoing automated search.

## REFERENCES

- [1] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.