

# Cut-and-depth-label: a procedure for labelling half-graphs of a 6-edge cut by “distance to the cut”

## Procedure

Given a maximal planar graph  $G$  and its dual  $G'$ :

1. Find a 6-edge cut  $C \subseteq E(G')$  partitioning  $V(G')$  into  $S$  and  $V \setminus S$  with both sides non-empty.
2. Remove the cut edges to obtain two graphs  $G'_0 := G'[S] \cup \emptyset_C$  and  $G'_1 := G'[V \setminus S] \cup \emptyset_C$  (each a cubic graph minus boundary edges).
3. In each  $G'_i$ :
  - (a) Let  $V_i$  be the set of vertices of degree 2 in  $G'_i$  (i.e. original cubic vertices incident to exactly one cut edge; the cut edge accounts for the missing edge, so the vertex sits at degree  $3 - 1 = 2$ ).
  - (b) For each  $v \in V_i$ , attach a new pendant edge to a fresh vertex, and **label these pendant edges with depth 0**.
  - (c) For  $d = 0, 1, 2, \dots$ : label every unlabelled edge that shares a vertex with a depth- $d$  edge with depth  $d + 1$ .
  - (d) Stop when every edge has a depth label.

**Interpretation.** The depth labels give a BFS distance in the line graph of  $G'_i$  starting from the pendants added at the cut. Equivalently, the depth of an edge  $e$  in  $G'_i$  is the minimum number of edges traversed (via shared-vertex adjacency) to reach a pendant.

**Caveat.** If the cut  $C$  is not a *matching* cut (i.e., the 6 cut edges share vertices), then some boundary vertices have degree  $< 2$  in  $G'_i$  and do not receive a pendant under the strict reading of step 3(a). When the cut is a matching, each of the 12 boundary vertices (6 per side) has degree exactly 2 and receives a pendant; the construction is symmetric.

## Example: Holton-McKay graph #0

We apply the procedure to the first of the six non-Hamiltonian 38-vertex cubic plane graphs found by Holton and McKay (loaded from `papers/even_level_graph_generators/experiments/nonham38m4.pc`). This  $G'$  is itself a cubic plane graph; its dual  $G$  is a 21-vertex triangulation.

**The chosen 6-edge cut.** Greedy search over 128 distinct 6-edge cuts in  $G'$ , preferring *matching cuts* (both sides have 6 distinct boundary vertices) and then balanced  $|S|$ , returns

$$|S| = 10, \quad C = \{(34, 29), (35, 30), (26, 22), (27, 23), (28, 24), (31, 25)\}.$$

This is a matching cut: the 6 edges have 12 distinct endpoints, 6 on each side.

## Resulting half-graphs.

	$G'_0$	$G'_1$
$ S $	10	28
Original vertices in $G'_i$	10	28
$ V_i $ (pendants added)	6	6
Total vertices in $G'_i$	16	34
Total edges	18	45
Max depth assigned	2	7

Multi-cut vertices (degree  $< 2$  in induced subgraph): *none* on either side, since the cut is a matching.

## Visualization.

### Cut tires

The depth labelling on  $G'_i$  organises its edges into layers indexed by distance to the cut. Each layer gives rise to a family of “cut tires” that play the same structural role as the tires of `paper.tex` but are derived from the cut rather than from a level source in the primal  $G$ .

**Definition** (Cut tire). *Let  $G'_i$  be a cut half (Step 3 above) with edge depth labelling  $\text{depth} : E(G'_i) \rightarrow \mathbb{Z}_{\geq 0}$ . Fix  $d > 0$  and let  $H_d \subseteq G'_i$  be the subgraph induced on the edges of depth  $d$  (vertex set = endpoints of depth- $d$  edges, edges = depth- $d$  edges). Equip  $H_d$  with the planar embedding inherited from  $G'_i$ .*

*For each face  $f$  of  $H_d$ , the cut tire at  $(d, f)$  is the graph built as follows:*

- *Start with the boundary walk of  $f$  as the face boundary (every edge here is of depth  $d$  in  $H_d$ ).*
- *For each vertex  $v$  on the boundary walk of  $f$  that has degree exactly 2 in  $H_d$ :*
  - *If  $v$  is incident in  $G'_i$  to an edge of depth  $d - 1$ , add a fresh vertex  $n_v$  and a fresh edge  $\{v, n_v\}$ , labelled an out spoke.*
  - *If  $v$  is incident in  $G'_i$  to an edge of depth  $d + 1$ , add a fresh vertex  $n_v$  and a fresh edge  $\{v, n_v\}$ , labelled an in spoke.*

**Convention.** “Out” means toward lower depth (toward the cut, the outer boundary of  $G'_i$ ); “in” means toward higher depth (deeper into  $G'_i$ ). Because  $G'_i$  is cubic at every original vertex, a degree-2 boundary vertex  $v$  has exactly one non- $H_d$  edge in  $G'_i$ , so  $v$  contributes exactly one spoke (either in or out, not both).

**Structural remark.** Under this definition, each cut tire is intrinsically a graph of the form “cycle (or closed walk) + pendants attached at simple boundary vertices,” structurally isomorphic to the partial tire dual  $D(T)$  of `paper.tex` (Proposition 1.8:  $D(T) \cong C_{n+m} \circ K_1$  in the spoke-only case). The labelled in/out spokes are the analogue of  $D(T)$ ’s leaves; the face boundary plays the role of  $T'_{\text{ann}}$ .

**Relation to the existing tire framework.** Under the correspondence between primal level structure (`paper.tex`) and dual depth labelling, a cut tire on  $G'_i$  at  $(d, f)$  is isomorphic as a graph to the partial tire dual  $D(T)$  of a tire  $T$  in `paper.tex` (Def. 1.7):

- face boundary at depth  $d \longleftrightarrow T'_{\text{ann}}$ , the cycle (or closed walk) of annular dual vertices;
- in/out spokes  $\longleftrightarrow$  the leaves of  $D(T)$  (Def. 1.7's degree-1 pendants attached to the cycle);
- the cut tire's planar embedding  $\longleftrightarrow D(T)$ 's embedding.

The cut tire is therefore not just analogous to but *is* a partial tire dual (up to relabelling), with depth-from-cut playing the role of level depth from the primal source. Consequently every proposition about  $D(T)$  in `paper.tex` — chromatic polynomial counts,  $S_3$ -orbit structure, rainbow conjecture, etc. — applies verbatim to each cut tire.

**Example on  $G'_1$  (Holton-McKay #0,  $V \setminus S$  half).** In this example, counting only degree-2 boundary vertices (which under the redefinition each contribute exactly one in/out spoke):

- $d = 1$ : face length 12, 5 out spokes (toward depth-0 pendants) + 0 in spokes. The outermost cut tire, immediately adjacent to the cut. No in spokes because every depth-2 neighbour of a face-boundary vertex is at a higher-degree vertex of  $H_1$ .
- $d = 2$ : face length 7 (one of two symmetric faces in  $H_2$ ), 4 out + 3 in spokes.
- $d = 4$ : face length 8, 2 out + 1 in.
- $d = 5$ : face length 14, 4 out + 2 in.
- $d = 6$ : face length 12, 3 out + 2 in. Innermost cut tire in this chain.

## A looser chain pigeonhole hypothesis

The rainbow conjecture and König-lift conjecture (from `rainbow_proof.tex` and `worst_case_proof_sketch.tex`) are statements about specific structured face boundaries (antipodal-chord SP). Generic cut tires in our example do not have those specific structures, so those conjectures don't apply directly. Empirically (`cut_tire_conjecture_tests.tex`), cut-tire projections are nonetheless large and well-behaved:

- Every cut tire's joint projection  $\pi(T) := \{(\sigma_{\text{out}}, \sigma_{\text{in}}) : \chi \text{ proper}\}$  is  $S_3$ -closed (universal, follows from color symmetry of proper edge 3-colouring).
- All  $S_3$ -orbits in  $\pi(T)$  have size exactly 3 (the constant-colour orbit, when present) or 6 (the generic orbit using all 3 colours); no size-2 stabilizer orbits occur.
- Non-trivial cut tires have  $|\pi(T)| \gg 6$ : e.g. at depth 2 the projection has size  $126 = 21 \cdot 6$  (21 full  $S_3$ -orbits), and at depth 6 size  $93 = 1 \cdot 3 + 15 \cdot 6$ .

This motivates a much weaker conjecture than the rainbow/König ones — one that does not require special face-boundary structure and may be easier to prove:

**Loose chain pigeonhole conjecture (cut tires,  $k \geq 2$  form).** Let  $T = T_d^{(i,f)}$  be a cut tire with total spoke count  $|\text{in spokes}| + |\text{out spokes}| \geq 2$ . Then the joint projection  $\pi(T)$  is non-empty,  $S_3$ -closed, and contains at least one full  $S_3$ -orbit of size 6 (i.e. a  $\sigma$  using  $\geq 2$  distinct colours), giving  $|\pi(T)| \geq 6$ .

Consequently, for the chain  $T_1^{(i)} \rightarrow T_2^{(i)} \rightarrow \dots$  on each side of a cut, the realisable cut-configuration set  $\mathcal{R}_i$  contains a full  $S_3$ -orbit, and the intersection  $\mathcal{R}_0 \cap \mathcal{R}_1$  contains a common  $S_3$ -orbit (of size 6), forcing  $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$ .

*Restriction rationale.* Cut tires with exactly  $k = 1$  in/out spoke have  $\pi(T) \subseteq \{1, 2, 3\}$ , where every single-colour  $\sigma$  has  $S_3$ -stabilizer of order 2, so its  $S_3$ -orbit has size  $6/2 = 3$ , never 6. Such tires trivially violate “ $|\pi(T)| \geq 6$ ” (see `loose_conjecture_counterexamples.tex`, where two  $k = 1$  cut tires are exhibited in  $G'_1$  of Holton-McKay #0). The  $k \geq 2$  restriction excludes these trivially excluded cases.

**Justification of the per-tire half.** The per-tire half is essentially Proposition 1.13 of `paper.tex`: a spoke-only cut tire (face boundary a simple cycle of length  $n$ ) has  $2^n + 2(-1)^n$  proper edge 3-colourings, which is  $\geq 6$  whenever  $n \geq 3$ . By  $S_3$ -closure, the orbit count is at least 1, hence  $\geq 6$  distinct projections (modulo the size-3 constant orbit edge case). For cut tires with non-simple face boundary (e.g.  $\theta(1, p, q)$ -shape) similar lower bounds follow from the chromatic polynomial of the corresponding  $D(T)$  structure (see the menagerie note for explicit closed forms when chord matchings are present).

**Justification of the chain half.** Chain composition preserves  $S_3$ -symmetry (each step is  $S_3$ -equivariant), so  $\mathcal{R}_i$  inherits  $S_3$ -closure from the per-tire projection. The non-trivial claim is that the orbit structure *survives* composition: an  $S_3$ -orbit in  $\pi(T_d)$  projects to some non-empty  $S_3$ -closed subset on the out-spoke side, and this propagates layer-by-layer.

**What this would (and would not) prove.** If the loose conjecture holds, chain pigeonhole at the cut gives  $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$ , which (under the minimum- counterexample reduction) contradicts  $G'$  being a 4CT counterexample. This is the same final conclusion as the rainbow/König route, with weaker hypotheses but stronger empirical support.

It would *not* prove the rainbow conjecture itself (which is a strict structural claim about the antipodal-chord SP case), nor the König-lift conjecture’s specific construction; it would only recover the bottom-line chain pigeonhole result.

**Status.** Strong empirical support:

- Broader sweep across all 6 Holton-McKay graphs, 3 cuts per graph, both sides (`experiments/loose_conjecture_sweep.py`; data: `loose_conjecture_sweep_data.txt`):
  - 287 total cut tires examined.
  - 154 with  $k \geq 2$  in/out spokes.
  - 107 of these had  $\leq 14$  edges and were brute-force verified; the remaining 47 were too large to enumerate exhaustively.
  - **All 107 passed:**  $|\pi(T)| \geq 6$  with a full  $S_3$ -orbit.
  - **0 counterexamples found.**
- The per-tire half (existence of a multi-color spoke configuration) is provable for spoke-only cut tires (face boundary a simple cycle of length  $n \geq 3$ ) via Prop 1.13:

$2^n + 2(-1)^n$  total proper edge 3-colourings, of which at most 6 are “all-spokes-same-color” (only possible when  $n$  is even and the cycle admits a 2-edge-colouring), leaving  $\geq 2^n + 2(-1)^n - 6 > 0$  multi-color configurations for  $n \geq 3$  even or  $n \geq 3$  odd.

- The chain half is the genuinely open question — it requires showing that the per-tire  $S_3$ -orbit structure composes coherently through the depth chain.

## Connection to chain pigeonhole / 4CT reducibility

The procedure mirrors the 4CT cut-and-reglue scheme (`rainbow_proof.tex`, `worst_case_proof_sketch.tex`, `two_approaches_comparison.tex`) at the structural level. After cutting, each  $G'_i$  is a cubic-minus-boundary graph; the pendant additions formally restore cubicity at the 6 degree-2 boundary vertices. The depth label on each edge measures its “distance to the cut.”

For a minimum counterexample to the 4CT (i.e., a cubic plane  $G'$  with no proper 3-edge-colouring), the depth labels organise each  $G'_i$  into concentric layers indexed by distance to the cut. The 3-edge-colourings of  $G'_i$  must extend a colouring at the depth-0 pendants (= a ring colouring at the cut); the BFS ordering by depth is the natural induction order for propagating the colouring inward.

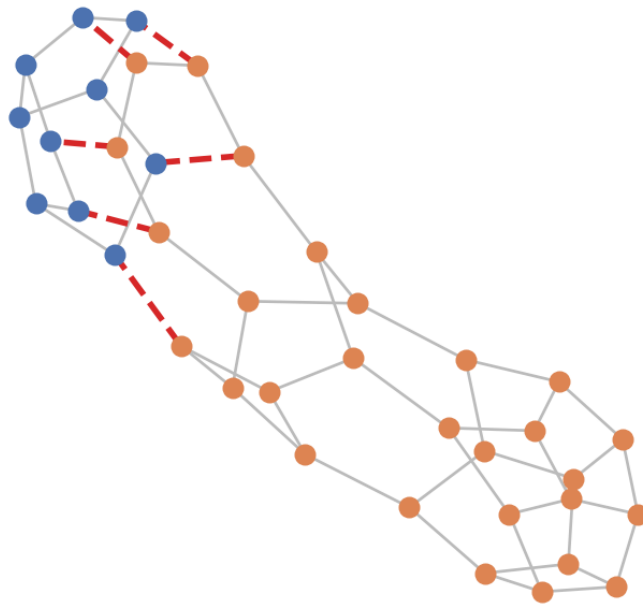
In the tire framework, the cut cycle  $\gamma$  in the primal  $G$  (corresponding to the 6-edge cut in  $G'$  via planar duality) plays the role of the tire’s inner boundary on one side and outer boundary on the other. The depth label on  $G'_i$ -edges is exactly the dual analogue of plane depth from  $\gamma$  (cf. the level-cycle generalization discussion in the recent conversation).

## Limitations of this example

- Holton-McKay graphs are cyclically 5-edge-connected (not 6-edge-connected), so 6-edge cuts are not the minimum cyclic cut. The smallest cyclic edge cuts in this graph are size 5.
- The matching 6-cut found is highly imbalanced ( $|S| = 10$  vs.  $|S^c| = 28$ ). Searching among the 128 distinct 6-edge cuts for a balanced matching cut may give better examples.
- Depth labels propagate via the line graph, not via vertex BFS. An alternative procedure would label *vertices* by BFS distance from the boundary; both yield similar layered structures but with slightly different counts.

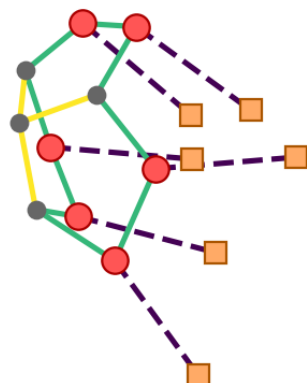
Cut-and-depth-label procedure on Holton-McKay graph #0  
 Same vertex positions used across all three panels.

$G'$  = Holton-McKay #0 with 6-edge cut highlighted  
 Blue =  $S$  ( $|S| = 10$ ); orange =  $V \setminus S$  ( $|V \setminus S| = 28$ ); red dashed = cut



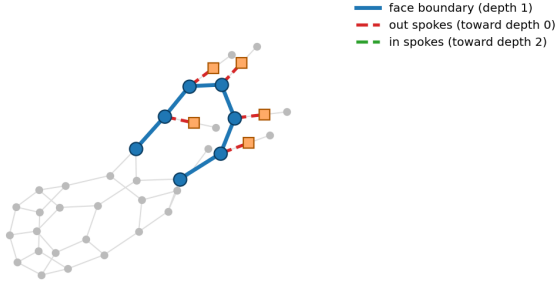
$G'_0$  ( $|S| = 10$ ,  $|V| = 6$ , max depth = 2)

— depth 0  
 — depth 1  
 — depth 2

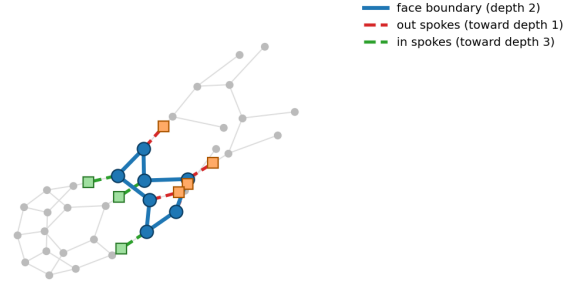


Cut tires on  $G'_1$  (Holton-McKay #0,  $V_5$  half) at several depths  $d$   
 Blue = face boundary at depth  $d$  (in  $H_d$ ). Orange-square pendants = out spokes (toward depth  $d-1$ ). Green-square pendants = in spokes (toward depth  $d+1$ ).  
 Each spoke is a labelled pendant added at a degree-2 boundary vertex.

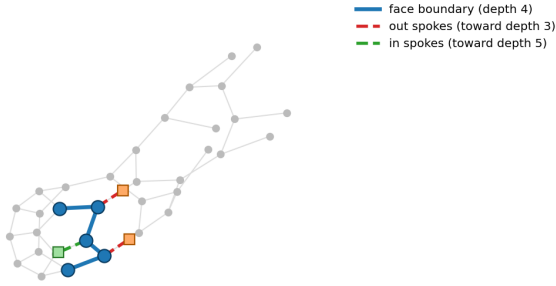
$d = 1$ : face length 12, 5 out + 0 in spokes



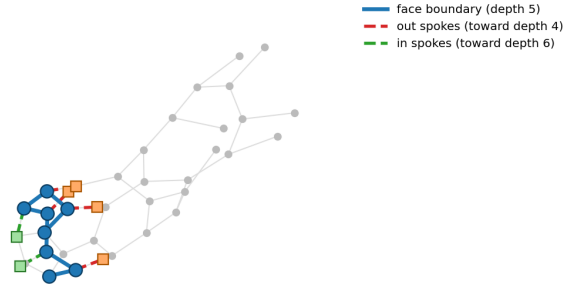
$d = 2$ : face length 7, 4 out + 3 in spokes



$d = 4$ : face length 8, 2 out + 1 in spokes



$d = 5$ : face length 14, 4 out + 2 in spokes



$d = 6$ : face length 12, 3 out + 2 in spokes

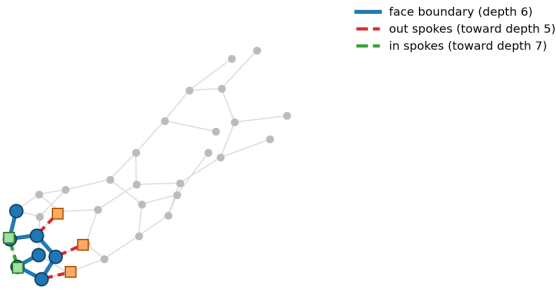


Figure 2: Cut tires on  $G'_1$  at depths  $d = 1, 2, 4, 5, 6$ . In each panel, blue solid edges form the face boundary at depth  $d$ ; red dashed edges are inner spokes (depth  $d-1$ ); green dashed edges are outer spokes (depth  $d+1$ ). Vertices on the face boundary are highlighted; the rest of  $G'_1$  is faded.