PLANE DIAMOND COLORING

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Abstract.

Notation

For a coloring $C : V(G) \to S$ and a color $c \in S$, we write $C^{-1}(c) = \{v \in V(G) : C(v) = c\}$ for the preimage of c under C, i.e., the color class of c.

1. Definitions

Definition 1.1. Let G be a graph and let $u \in V(G)$. The *distance partition* of G from u is the partition $\{L_0, L_1, L_2, \dots\}$ of $V(G)$ obtained by breadth-first search from u:

$$L_0 = \{u\}, \qquad L_{i+1} = \{v \in V(G) \setminus (L_0 \cup \cdots \cup L_i) : v \text{ is adjacent to some } w \in L_i\}.$$

Equivalently, $L_i = \{v \in V(G) : d(v,u) = i\}$, where $d(v,u)$ denotes the graph distance between v and u in G. We call each L_i the *i-th level* of the partition.

Definition 1.2. Let G be a maximal planar graph with a plane embedding, and let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from some $u \in V(G)$. The *diamond scaffold* of G relative to u is the spanning subgraph $G^\diamond \subseteq G$ obtained by removing every edge $\{x, y\} \in E(G)$ such that $x, y \in L_i$ for some i.

Definition 1.3. Let G be a maximal planar graph. A *plane diamond coloring* of G is a proper 4-coloring C of G such that there exist two colors c_a, c_b and a diamond scaffold $G^\diamond$ of G with a proper 2-coloring $C^\diamond : V(G) \to \{c_a, c_b\}$ satisfying

$$C^\diamond(v) = c_a \quad \text{for every } v \in C^{-1}(c_a),$$
$$C^\diamond(v) = c_b \quad \text{for every } v \in C^{-1}(c_b).$$

2. Results

Theorem 2.1. *The diamond scaffold of any maximal planar graph G is 2-colorable.*

Proof. Let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from the chosen vertex u, and let $G^\diamond$ be the resulting diamond scaffold. We show $G^\diamond$ is bipartite by exhibiting a proper 2-coloring.

For any edge $\{x, y\} \in E(G)$, the depths of x and y differ by at most 1: if $x \in L_i$, then prepending the edge $\{y, x\}$ to a shortest path from x to u gives a walk of length $i+1$ from y to u, so $y \in L_j$ for some $j \leq i+1$, and symmetrically $i \leq j+1$. Hence $|i - j| \leq 1$.

By construction, $G^\diamond$ contains no edge with both endpoints in the same level L_i. Combined with the bound above, every edge of $G^\diamond$ joins some L_i to L_{i+1}. Color

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each vertex $v \in L_i$ by the parity of i. Every edge of $G^\diamond$ connects vertices of opposite parity, so this is a proper 2-coloring. □

Lemma 2.2. *The diamond scaffold $G^\diamond$ of a maximal planar graph G relative to u is connected.*

Proof. Let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from u. We show by induction on i that every vertex of L_i is connected to u in $G^\diamond$. The base case $i = 0$ is immediate, since $L_0 = \{u\}$. For $i \geq 1$, let $v \in L_i$. By definition of L_i, there is a shortest path from v to u of length i in G, whose penultimate vertex w lies in L_{i-1}. The edge $\{v, w\}$ joins L_i to L_{i-1}, hence is not a level edge, hence belongs to $G^\diamond$. By the inductive hypothesis w is connected to u in $G^\diamond$, so v is as well. □

Proposition 2.3. *A maximal planar graph G has a plane diamond coloring if and only if there exist a proper 4-coloring C of G, a vertex $u \in V(G)$, and two distinct colors c_a, c_b such that, with respect to the distance partition $\{L_0, L_1, L_2, \dots\}$ of G from u,*

$$C^{-1}(c_a) \subseteq \bigcup_{i \text{ even}} L_i \qquad \text{and} \qquad C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i.$$

Proof. Since $G^\diamond$ is connected and bipartite (Theorem 2.1 and Lemma 2.2), its proper 2-coloring is unique up to swapping the two colors, and is given by the parity of level. Hence a proper 2-coloring $C^\diamond : V(G) \to \{c_a, c_b\}$ of $G^\diamond$ exists with $C^\diamond(v) = c_a$ on the even-parity layers and $C^\diamond(v) = c_b$ on the odd-parity layers (or vice versa). The agreement condition $C(v) = C^\diamond(v)$ on $C^{-1}(c_a) \cup C^{-1}(c_b)$ is then equivalent to the stated containment. □

Remark 2.4. The conjecture below asserts a structural property of 4-colorings of maximal planar graphs strictly stronger than the conclusion of the Four Color Theorem [1, 2]: it requires not merely the existence of a proper 4-coloring, but the existence of a proper 4-coloring together with a root u such that two of the four color classes are separated by the parity of the BFS layering from u.

Conjecture 2.5. *Every maximal planar graph G has a plane diamond coloring.*

Theorem 2.6. *Conjecture 2.5 is false. Moreover, the smallest counterexample has order* 13*, and is unique up to isomorphism among triangulations of order at most* 13.

Proof. Let G be the maximal planar graph on 13 vertices with graph6 string[1]

`LQhO?SFIwnTYTi`

shown in Figure 1. Equivalently, G has edge set

$$\begin{aligned}
&\{0,2\}, \{0,4\}, \{0,11\}, \{1,3\}, \{1,5\}, \{1,12\}, \{2,4\}, \{2,9\}, \{2,11\},\\
&\{3,5\}, \{3,10\}, \{3,12\}, \{4,7\}, \{4,9\}, \{4,11\}, \{5,8\}, \{5,10\}, \{5,12\},\\
&\{6,7\}, \{6,8\}, \{6,9\}, \{6,10\}, \{6,11\}, \{6,12\}, \{7,8\}, \{7,9\}, \{7,10\},\\
&\{7,11\}, \{8,9\}, \{8,10\}, \{8,12\}, \{9,11\}, \{10,12\}.
\end{aligned}$$

We have $|V(G)| = 13$ and $|E(G)| = 33 = 3 \cdot 13 - 6$, so G is a triangulation.

By Proposition 2.4, it suffices to verify that for every choice of root $u \in V(G)$ and every pair of distinct colors c_a, c_b, no proper 4-coloring C of G satisfies $C^{-1}(c_a) \subseteq$

[1] We use the standard graph6 encoding of McKay; see [3].

$\bigcup_{i \text{ even}} L_i^{(u)}$ and $C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i^{(u)}$, where $\{L_i^{(u)}\}$ denotes the distance partition from u.

For a fixed root u, the existence of such a 4-coloring C together with colors c_a, c_b is equivalent to 4-colorability of the auxiliary graph H_u obtained from G by adjoining two new vertices α, β, joining α to every vertex in $\bigcup_{i \text{ odd}} L_i^{(u)}$, joining β to every vertex in $\bigcup_{i \text{ even}} L_i^{(u)}$, and adding the edge $\{\alpha, \beta\}$. Indeed, in any proper 4-coloring of H_u the colors of α and β are distinct and absent from the odd-parity and even-parity layers of G respectively, yielding the required colors $c_a := C(\alpha)$ and $c_b := C(\beta)$. Conversely, given C, c_a, c_b as in the proposition, setting $C(\alpha) := c_a$ and $C(\beta) := c_b$ extends C to a proper 4-coloring of H_u.

A direct computation (using `Sage`'s `chromatic_number`) verifies that $\chi(H_u) > 4$ for every $u \in V(G)$, so G admits no plane diamond coloring.

For minimality and uniqueness, we exhaustively enumerated every maximal planar graph of order at most 13 using `Sage`'s `graphs.planar_graphs` generator (with `minimum_connectivity=3` and `maximum_face_size=3`). The numbers of triangulations $1, 1, 2, 5, 14, 50, 233, 1249, 7595, 49566$ at orders $4, 5, \ldots, 13$ respectively (matching OEIS A000109) were each tested for the existence of a plane diamond coloring, and exactly one — the graph G above, occurring at order 13 — was found to lack one. $\square$

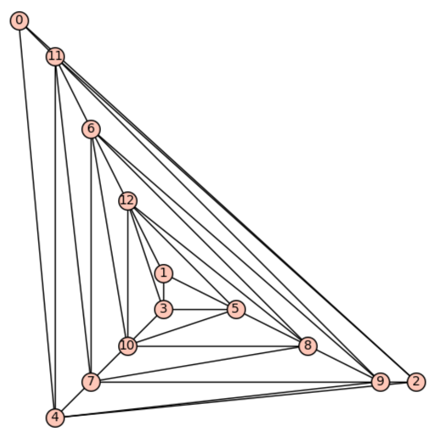

FIGURE 1. The unique smallest maximal planar graph with no plane diamond coloring; it has 13 vertices and degree sequence $(6, 6, 6, 6, 6, 6, 6, 5, 5, 4, 4, 3, 3)$.

REFERENCES

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