

# LEVEL RESOLUTIONS OF MAXIMAL PLANAR GRAPHS

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**ABSTRACT.** We propose a structural reformulation of the four color theorem in terms of *level resolutions* of maximal planar graphs. A level structure on a plane graph  $G$  is defined by BFS from a chosen level source (either a face or a degree-3 vertex), partitioning vertices into levels. A triangulation  $G'$  on the same vertex set is a *level resolution* of  $G$  from this source if the subgraphs of  $G'$  induced by even- and odd-level vertices are both bipartite. By construction, any level resolution admits an explicit 4-coloring obtained by 2-coloring each parity subgraph independently. The structural foundation of this approach is that each level subgraph  $L_k$  of  $G$  is outerplanar, and outerplanar graphs are 3-chromatic; the level-resolution problem is precisely to flip edges of  $G$  to reduce each  $L_k$  from chromatic number 3 to 2. We present computational results characterizing which isomorphism classes of maximal planar graphs on  $n = 6, \dots, 11$  vertices arise as level resolutions, and verify that every iso-class is reachable at every tested size.

## 1. INTRODUCTION

The four color theorem (4CT) asserts that every planar graph is 4-colorable. Equivalently, every maximal planar graph (triangulation) is 4-colorable. The Appel–Haken proof [1] and subsequent Robertson–Sanders–Seymour–Thomas refinement [2] rely on discharging arguments and computer-verified reducible configurations. Human-readable proofs remain elusive.

We propose a different structural approach. Given a plane triangulation  $G$  and a choice of *level source*, BFS from the source partitions the vertices into levels. A triangulation  $G'$  on the same vertex set is a *level resolution* of  $G$  if, when its vertices are labelled by the parity of their  $G$ -levels, the subgraph of  $G'$  induced by even-parity vertices and the subgraph induced by odd-parity vertices are both bipartite. The 4-coloring of  $G'$  then follows by definition: 2-color each parity subgraph and identify the four resulting classes with four distinct colors.

The remaining question is when level resolutions exist. We conjecture:

- (i) every plane triangulation  $G'$  is a level resolution of some plane triangulation  $G$  via some level source; or, in a restricted form,
- (ii) every plane triangulation of minimum degree at least 4 is a level resolution of some plane triangulation.

This paper formalizes the definitions and presents computational evidence bearing on (i)–(ii) for small vertex counts.

## 2. DEFINITIONS

Throughout,  $G = (V, E)$  is a plane maximal planar graph (a triangulation) with a fixed planar embedding  $\Pi_G$ . We write  $|V| = n$ , so  $|E| = 3n - 6$  and  $G$  has  $2n - 4$  triangular faces.

**Definition 2.1** (Level source). A *level source* of  $G$  is either:

- a face  $F$  of  $G$  (all vertices of  $F$  are level-0 sources), or
- a vertex  $v$  of degree 3 (the singleton  $\{v\}$  is a level-0 source).

**Definition 2.2** (Levels). Given a level source  $S \subseteq V$ , the *level* of  $v \in V$  is  $\ell_G(v) = \text{dist}_G(v, S)$ , the graph distance from  $v$  to the nearest source vertex.

**Definition 2.3** (Parity subgraph). Let  $G$  be a triangulation with level source  $S$ , and let  $G'$  be a triangulation on the same vertex set as  $G$ . The *even parity subgraph*  $E_{G,S}(G')$  is the subgraph of  $G'$  induced by  $\{v \in V : \ell_G(v) \equiv 0 \pmod{2}\}$ . The *odd parity subgraph* is defined analogously for odd  $\ell_G$ .

**Definition 2.4** (Level resolution). A triangulation  $G'$  on the same vertex set as  $G$  is a *level resolution* of  $G$  from level source  $S$  if both the even and odd parity subgraphs  $E_{G,S}(G')$  and  $O_{G,S}(G')$  are bipartite.

By construction, when  $G'$  is a level resolution of  $G$  via  $S$ , an explicit proper 4-coloring of  $G'$  is obtained by 2-coloring each parity subgraph independently (e.g., via BFS) and assigning the four resulting classes to distinct colors: even vertices receive red/blue, odd vertices receive yellow/green. The edges of  $G'$  partition into (i) edges within a parity subgraph, properly colored by the bipartition of that subgraph; and (ii) edges between an even-parity and odd-parity vertex, which connect disjoint color sets and so are properly colored.

## 3. STRUCTURAL FOUNDATION: OUTERPLANARITY OF LEVEL SUBGRAPHS

For each integer  $k \geq 0$  and each  $(G, S)$ , write  $L_k$  for the subgraph of  $G$  induced by the level- $k$  vertices.

**Theorem 3.1.** *For every plane triangulation  $G$  and every level source  $S$  of  $G$ , each level subgraph  $L_k$  is outerplanar.*

*Proof.* For  $k = 0$ ,  $L_0$  is either a single vertex (when  $S$  is a degree-3 vertex) or the triangle bounding the source face (when  $S$  is a face), both outerplanar. Fix  $k \geq 1$  and suppose, for contradiction, that  $L_k$  is not outerplanar.

Let  $D_k$  denote the planar drawing of  $L_k$  inherited from  $\Pi_G$ : that is, the set of points and curves in the plane representing the vertices and edges of  $L_k$  exactly as they appear in the embedding  $\Pi_G$ . Since  $L_k$  is not outerplanar, no face of  $D_k$  has every vertex of  $L_k$  on its boundary.

Let  $F^*$  be the face of  $D_k$  containing the source: when  $S = \{v\}$ , the face containing the point  $v$ ; when  $S$  is a face  $F$  of  $G$ , the face containing the open region of  $F$  together with its three bounding vertices. The latter is well defined because each vertex of  $F$  lies at level 0 (hence is not a vertex of  $L_k$ ) and each edge of  $F$  joins two level-0 vertices (hence is not an edge of  $L_k$ ), so  $F$  and its boundary lie in a single component of  $\mathbb{R}^2 \setminus D_k$ . By assumption there exists  $u \in L_k$  with  $u \notin \partial F^*$ .

Choose a BFS path  $P : v_0, v_1, \dots, v_k = u$  with  $v_0 \in S$  and  $v_i \in L_i$ . For  $0 \leq i \leq k - 1$ ,  $v_i$  lies in  $L_i$  and so is not a vertex of  $L_k$ ; for  $1 \leq i \leq k$ , the edge

$v_{i-1}v_i$  joins  $L_{i-1}$  to  $L_i$  and so is not an edge of  $L_k$ . Hence, viewed as a curve in the plane,  $P$  meets the drawing  $D_k$  only at its endpoint  $u$ .

The complement  $\mathbb{R}^2 \setminus D_k$  is open, and  $P \setminus \{u\}$  is its continuous image of a connected set, hence lies in a single face of  $D_k$ . Since  $v_0 \in F^*$ , in fact  $P \setminus \{u\} \subseteq F^*$ , so  $u \in \overline{F^*}$  and therefore  $u \in \partial F^*$ , contradicting the choice of  $u$ .  $\square$

The combinatorial significance of Theorem 3.1 is that outerplanar graphs are 3-chromatic [4]: their chromatic number is at most 3. Hence each  $L_k$  admits an independent 3-coloring, giving an immediate (but suboptimal) coloring of  $G$  using at most  $3 \cdot \text{depth}(G, S)$  colors when levels are colored independently. To recover a 4-coloring of  $G'$  via the parity-2-coloring strategy, what is required is to reduce each  $L_k$ 's chromatic number from 3 to 2, equivalently to remove every odd cycle from each  $L_k$ :

**Proposition 3.2.** *If  $G'$  is a triangulation on the same vertex set as  $G$  such that for every  $k$ , the subgraph of  $G'$  induced by the level- $k$  vertices of  $(G, S)$  is bipartite, and  $G'$  contains no edge between vertices at  $G$ -levels of equal parity and differing by exactly 2, then  $G'$  is a level resolution of  $G$  via  $S$ .*

*Proof.* The even parity subgraph  $E_{G,S}(G')$  is the disjoint union of the even-level subgraphs of  $G'$  (since by hypothesis no edge of  $G'$  joins two even levels), each of which is bipartite. A disjoint union of bipartite graphs is bipartite. The same argument applies to the odd parity subgraph.  $\square$

This is the form of level resolution we seek to realize constructively: flips applied to  $G$  that break every odd cycle in every  $L_k$  without introducing cross-parity edges of distance 2.

#### 4. THE FOUR-COLOR CONJECTURE VIA LEVEL RESOLUTIONS

**Conjecture 4.1** (Resolution preimage). Every plane triangulation  $G'$  on  $n$  vertices is a level resolution of some plane triangulation  $G$  on  $n$  vertices.

If Conjecture 4.1 holds, the 4-coloring of any triangulation  $G'$  follows from the definition: exhibit a level-resolution preimage  $G$ , compute the BFS levels in  $G$  from the witness source, and 4-color  $G'$  via the parity 2-coloring.

#### 5. COMPUTATIONAL EVIDENCE

We enumerated all non-isomorphic triangulations on  $n \in \{6, \dots, 11\}$  via vertex insertion followed by edge-flip closure (see `triangulation_gen.py` and the faster `triangulation_gen_fast.py` for  $n \geq 11$ ). For each isomorphism class, we computed the full set of iso-classes reachable as level resolutions across all valid level sources.

**5.1. Coverage at  $n = 6, \dots, 11$ .** Table 1 lists the resolution behavior for each iso-class. A class  $T_i$  is *covered* if it appears as the resolution iso-class of some triangulation.

**Observation 5.1.** For every  $n \in \{6, \dots, 11\}$ , every plane-triangulation iso-class on  $n$  vertices is a level resolution of some plane triangulation on the same vertex set.

$n$	Iso-classes	Reachable as level resolutions
6	2	all 2
7	5	all 5
8	14	all 14
9	50	all 50
10	233	all 233
11	1249	all 1249

TABLE 1. Iso-class coverage under the level-resolution definition.

Equivalence to 4-colorability. A 2-partition  $V = V_0 \sqcup V_1$  for which both  $G'[V_0]$  and  $G'[V_1]$  are bipartite induces a proper 4-coloring of  $G'$  (combine the bipartition of  $V_0$  into colors  $\{R, B\}$  and that of  $V_1$  into  $\{Y, G\}$ ), and conversely, any proper 4-coloring grouped pairwise produces such a partition. Hence by Definition 2.4,  $G'$  is a level resolution of some  $(G, S)$  if and only if  $G'$  admits a bipartite 2-partition of cardinality realizable as  $(|V_e|, |V_o|)$  for some level source. Surjectivity at a given  $n$  is therefore equivalent to 4-colorability of every triangulation on  $n$  vertices together with realizability of the partition cardinality by some BFS. Our computational verification of Observation 5.1 does not invoke 4CT: we enumerate vertex partitions directly and check bipartiteness of the induced subgraphs.

**5.2. Surjectivity at  $n = 12$ : the icosahedron.** The icosahedron is the unique 5-regular triangulation on 12 vertices and a natural test case at  $n = 12$  since it has no degree-3 vertex (so the  $\text{md}_4$  restriction is irrelevant) and high symmetry constrains the achievable parity-cardinality splits to  $(6, 6)$  from any source. We verify directly that the icosahedron admits a bipartite 2-partition of cardinality  $(6, 6)$ : with vertices labelled as in the standard icosahedral graph, the partition  $\{0, 1, 2, 3, 4, 7\} \mid \{5, 6, 8, 9, 10, 11\}$  has both classes inducing bipartite subgraphs (each is a 6-cycle). By Definition 2.4, the icosahedron is therefore a level resolution of some plane triangulation on 12 vertices.

**Observation 5.2.** The icosahedron is a level resolution of some plane triangulation on 12 vertices.

**5.3. Restatement of the resolution-preimage conjecture.** In light of Observations 5.1 and 5.2, we restate Conjecture 4.1 more confidently:

**Conjecture 5.3** ( $\text{md}_4$  surjectivity). For every  $n \geq 6$ , every minimum-degree-4 plane triangulation on  $n$  vertices is a level resolution of some plane triangulation on  $n$  vertices.

By the equivalence noted in Section 3, this is equivalent to a 4-coloring statement: every minimum-degree-4 plane triangulation admits a proper 4-coloring whose color-class cardinalities, grouped pairwise, match some BFS-level parity cardinality on the same vertex set. Since the unrestricted preimage conjecture also appears to hold at every tested  $n$ , the  $\text{md}_4$  restriction may be unnecessary; we retain it here as the form most amenable to the constructive techniques explored in Section 6.

## 6. AN EDGE-FLIP RESOLUTION ALGORITHM

We describe an iterative edge-flip procedure aimed at producing, for a given  $(G, S)$ , a triangulation  $G'$  on the same vertex set whose simple level cycles (with respect to the  $G$ -levels from  $S$ ) are all even.

**6.1. Apex classification of  $L_k$ -edges.** Let  $k \geq 1$ . For each  $uv \in E(L_k)$ , the two triangles of  $G$  bounding  $uv$  have third vertices  $w, x$ , called the *apexes* of  $uv$ , with  $\ell_G(w), \ell_G(x) \in \{k-1, k, k+1\}$  by BFS. We call  $uv$  *intra-level* when  $\ell_G(w) = \ell_G(x) = k$ , and *cross-level* otherwise.

**Lemma 6.1.** *If both apexes of  $uv \in E(L_k)$  are at level  $k-1$ , then  $uv$  is a bridge of  $L_k$ .*

*Proof sketch.* In a plane triangulation, the neighbors of  $u$  in  $G$  at level  $\leq k-1$  form a contiguous arc in the cyclic order around  $u$ . If both apexes  $w, x$  of  $uv$  lie at level  $k-1$  on opposite sides of  $uv$ , then  $v$  lies in the complementary cyclic arc, which contains no other level- $k$  neighbor of  $u$ . The symmetric statement around  $v$  gives that  $u$  is  $v$ 's only level- $k$  neighbor in the corresponding arc, so  $uv$  is a bridge of  $L_k$ .  $\square$

In particular every edge on a cycle of  $L_k$  has at least one apex at level  $k$  or  $k+1$ .

**Proposition 6.2.** *Flipping  $uv \in E(L_k)$  with apexes  $w, x$  replaces  $uv$  with  $wx$  in  $G$ . The new edge  $wx$  belongs to  $L_k$  iff  $\ell_G(w) = \ell_G(x) = k$ , and to  $L_{k+1}$  iff  $\ell_G(w) = \ell_G(x) = k+1$ ; otherwise  $wx$  is cross-parity and lies in no level subgraph. In all cases  $uv$  is removed from  $L_k$ .*

**6.2. Facial depth and isolated faces.**

**Definition 6.3** (Facial depth). Let  $L_k$  be drawn with the outerplanar embedding inherited from  $\Pi_G$ , let  $D$  be the dual graph of this drawing with the outer face removed, and let  $\mathcal{B}$  be the set of inner faces incident to at least two edges of the outer face of  $L_k$ . The *facial depth* of an inner face  $F$  of  $L_k$  is

$$\text{depth}(F) = \min_{F' \in \mathcal{B}} \text{dist}_D(F, F'),$$

with the convention  $\text{depth}(F) = \infty$  if no such  $F'$  exists. An inner face is *isolated* if  $\text{depth}(F) \geq 1$ .

The seed set  $\mathcal{B}$  consists of the inner faces that already have two outer-face edges available as flip targets; Phase 2 below handles these directly. Phase 1 uses facial depth as a potential to push isolated odd faces toward  $\mathcal{B}$ .

A cycle that is *tricky-everywhere*, meaning every edge is intra-level, is necessarily isolated: an outer-face edge of  $L_k$  has a level- $(k-1)$  apex on its outer side and is therefore cross-level, so a tricky-everywhere cycle shares no edge with the outer face. Hence the tricky-everywhere cycles are a subset of the isolated odd cycles.

**6.3. Phase 1: interior faces. Procedure.** While some  $L_k$  contains an odd simple cycle whose corresponding inner face has facial depth  $\geq 1$  and shares no edge with the outer face, repeat:

- (1) compute facial depths for all simple level cycles of  $L_k$ ;
- (2) among interior odd faces (depth  $\geq 1$ , no outer-face edges) of maximum facial depth, pick one  $C$ ; even-parity interior faces of depth  $\geq 1$  may also be selected as  $C$ ;

- (3) find the inner face  $F'$  incident to  $C$  of minimum facial depth, and flip the edge shared between  $C$  and  $F'$ .

The restriction to faces with no outer-face edge in step (2) means that every edge of  $C$  borders another inner face, so a unique shared-edge flip target exists for each neighbor  $F'$ . The depth-guided choice of  $F'$  in step (3) progressively pushes the residual odd-face structure toward the seed set  $\mathcal{B}$  (depth 0). Even-face flips are optional restructuring moves that expand the reachable configuration space; the loop's termination is gated only by interior odd faces.

**6.4. Phase 2: outer-incident faces.** After Phase 1, every remaining odd simple cycle of  $L_k$  shares at least one edge with the outer face, whose apex pair includes a level- $(k-1)$  vertex and is therefore cross-level.

**Procedure.** For each  $L_k$ :

- every odd simple cycle  $C \subseteq L_k$  incident to the outer face *must* have exactly one of its outer-face edges flipped;
- every even simple cycle of  $L_k$  incident to the outer face *may* have at most one of its outer-face edges flipped (an optional restructuring move).

For the source-face level ( $k=0$  with face source  $S$ ), the  $L_0$  source triangle is itself an odd cycle whose three edges all bound the outer face; we treat  $L_0$  uniformly with higher levels, with the option of leaving the triangle intact when the resulting parity-subgraph configuration on  $G'$  permits.

Each flip is permitted even if the apex edge  $wx$  already exists in  $G$ , in which case  $G'$  is a multigraph; this does not affect bipartiteness of the parity subgraphs of  $G'$ , since a duplicated edge is bipartite-equivalent to a single edge.

#### 6.5. Simple level resolutions.

**Definition 6.4** (Simple level resolution). A plane triangulation  $G'$  is a *simple level resolution* of a plane triangulation  $G$  if there exists a level source  $S$  of  $G$  such that the algorithm of Sections 6–Phase 1 and Phase 2 applied to  $(G, S)$ , under some sequence of optional-move choices, produces  $G'$  as a simple-graph triangulation whose parity subgraphs are bipartite.

#### 6.6. Empirical status.

**Observation 6.5.** For every plane triangulation  $G$  on  $n \in \{9, 10, 11\}$  vertices, every level source  $S$ , and every  $k$  such that  $L_k$  contains an odd simple cycle, the algorithm produces a  $G'$  whose corresponding  $L_k$  is bipartite (in the underlying simple-graph view). Across the 29640 such  $(G, S, k)$  triples — 4645 at  $n \leq 10$  and 24995 at  $n = 11$  — Phase 1 always terminates and Phase 2 always succeeds.

Coverage test for Conjecture 6.7. For each  $n \in \{6, \dots, 11\}$  we enumerate all plane-triangulation iso-classes on  $n$  vertices. For each iso-class  $G$ , each level source  $S$  of  $G$ , and each branching choice within the algorithm — Phase 1 ties on which deepest interior face and which lowest-depth neighbor to flip, Phase 2 choices of which outer-face edge to flip for each odd or even outer-incident cycle (including the option to leave even cycles untouched), and the option to skip the source-triangle break when  $S$  is a face — we run Phase 1 to termination and then Phase 2, recording the algorithm's output  $G'$  as a labelled simple graph. We check three properties: (a)  $G'$  is a triangulation (no multi-edge survived the Phase 2 flips), (b) the parity subgraphs  $E_{G,S}(G')$  and  $O_{G,S}(G')$  are both bipartite, and (c) the

iso-class of  $G'$ . Aggregating over all  $(G, S, \text{branch-choices})$  triples yields the set of iso-classes attainable as algorithm outputs satisfying (a)+(b); we compare this set against the minimum-degree-4 iso-classes at each  $n$ .

**Observation 6.6.** For every  $n \in \{6, 7, 8, 9, 10, 11\}$ , every plane triangulation iso-class on  $n$  vertices with minimum degree at least 4 is a simple level resolution of some plane triangulation on  $n$  vertices. Concretely, the counts of minimum-degree-4 iso-classes — 1, 1, 2, 5, 12, 34 at  $n = 6, \dots, 11$  — are all reached by the algorithm with bipartite parity subgraphs (Definition 6.4).

**Conjecture 6.7** (Simple-resolution  $\text{md}_4$  surjectivity). For every  $n \geq 6$ , every minimum-degree-4 plane triangulation on  $n$  vertices is a simple level resolution of some plane triangulation on  $n$  vertices.

The minimum-degree-4 restriction in Conjecture 6.7 is necessary: at  $n = 8$ , the unique iso-class with three degree-3 vertices is not reachable by the algorithm; at  $n = 10$ , two further iso-classes with four degree-3 vertices and high-degree hubs fail to appear among algorithm outputs.

**Question 6.8.** Does Phase 1 terminate for all  $(G, S)$ ? Equivalently, is there an explicit monovariant on  $L_k$ 's face structure that strictly decreases on every Phase 1 flip?

## 7. DISCUSSION AND OPEN QUESTIONS

The computational results suggest the following:

- (1) Conjecture 4.1 (resolution preimage) holds at every tested size: all iso-classes on  $n \in \{6, \dots, 11\}$  vertices arise as level resolutions, and the icosahedron does at  $n = 12$  (Observations 5.1 and 5.2).
- (2) Each level subgraph  $L_k$  of  $G$  is outerplanar (Theorem 3.1), so each  $L_k$  is 3-chromatic classically and independently of 4CT. The level-resolution problem reduces to flipping edges of  $G$  so that each  $L_k$ 's chromatic number drops from 3 to 2, while avoiding creation of  $G$ -level-2 same-parity edges (Proposition 3.2).
- (3) Under Definition 2.4, being a level resolution is equivalent to admitting a proper 4-coloring whose color cardinalities group pairwise to a BFS-realizable parity split. The structural framing through outerplanarity refines this: a constructive 4-coloring of  $G'$  is obtained via independent 2-colorings of each  $L_k$  in  $G'$ , and the proof obligation is purely about removing odd cycles within outerplanar graphs by local edge flips, an operation that does not invoke 4CT.

The algorithm of Section 6 is the candidate constructive answer. Phase 1 iteratively flips the shared edge between the deepest interior odd face and its lowest-depth neighbor, pushing the residual odd-face structure toward the seed set  $\mathcal{B}$  at depth 0, with optional even-face restructuring moves along the way; Phase 2 disposes of the remaining outer-incident odd cycles by flipping an outer-face edge each (and optionally an even outer-incident face), accepting a multigraph if the apex edge already exists. Observation 6.5 records that the algorithm terminates and succeeds at the level-bipartiteness layer on all 29640 tested  $(G, S, k)$  triples at  $n \in \{9, 10, 11\}$ . Observation 6.6 records that every minimum-degree-4 iso-class on  $n \leq 11$  vertices is reached as a simple level resolution; Conjecture 6.7 extends this to all  $n$ . Question 6.8 asks whether Phase 1 terminates in general.

## 8. IMPLEMENTATION

The code accompanying this paper consists of the following modules:

- `level_cycles.py`: core library for levels, level cycles, flip candidates, and resolution enumeration.
- `triangulation_gen.py`: enumeration of all non-isomorphic triangulations on  $n$  vertices via vertex-insertion plus flip closure.
- `coverage.py`: iso-class coverage reports with optional source and target filters.
- `balanced_layout.py`: a planar drawing routine that starts from a Tutte embedding and uses random-search optimization to equalize interior face areas while maintaining planarity.
- `four_color.py`: 4-coloring of  $G'$  via independent BFS 2-colorings of parity subgraphs.
- Visualization scripts: `plot_oct.py`, `n7_examples.py`, `four_color_viz.py`.

## REFERENCES

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