A proof attempt at the worst-case overlap

Setup

Let γ be a cycle of length k (divisible by 3 for the "all-3" case), and let T_1, T_2 be two adjacent SP tires sharing γ with the threshold condition $|B^{(1)}_{\text{out}}| \geq k$ and $|B^{(2)}_{\text{in}}| \geq k$. Assume both tires' inner outerplanar graphs are configured so that every O-face has exactly 3 B_{in} edges (strict Latin case).

The empirical worst-case overlap is exactly $|S_3| = 6$ elements, forming a single S_3-orbit (see `tire_fiber_step2_large.tex`, Obs. on the S_3-orbit structure). This note sketches a proof.

What I can prove cleanly: both sides give γ-face partitions

Proposition (König $\Rightarrow$ overlap ≥ 6)**.** *Suppose both T_1 and T_2 induce* direct *all-3 face partitions $\mathcal{F}_1, \mathcal{F}_2$ of $E(\gamma)$. Then*

$$|\mathcal{L}(\gamma, \mathcal{F}_1) \cap \mathcal{L}(\gamma, \mathcal{F}_2)| \;\geq\; 6.$$

Proof sketch. Define a bipartite graph G:

- Left vertices: $\mathcal{F}_1$-faces (there are $k/3$ of them).
- Right vertices: $\mathcal{F}_2$-faces (also $k/3$).
- One edge from $F \in \mathcal{F}_1$ to $F' \in \mathcal{F}_2$ for each γ-edge $e \in F \cap F'$.

Since each γ-edge belongs to exactly one face of each partition, γ-edges biject with edges of G, and $|E(G)| = k$. Each face has 3 γ-edges, so every vertex of G has degree exactly 3. Thus G is 3-regular bipartite.

By König's edge-coloring theorem, every bipartite graph admits a proper Δ-edge-coloring; here $\Delta = 3$. So G has a proper 3-edge-coloring $\chi : E(G) \to \{1, 2, 3\}$. Transport χ back to γ-edges via the bijection: define $\sigma(e) := \chi(e)$. Then for any face $F \in \mathcal{F}_1$, the three γ-edges in F are the three G-edges incident to the F-vertex (degree 3), and χ being proper forces these to use all three colors — so $\sigma|_F$ is a permutation of $\{1, 2, 3\}$. Same for $\mathcal{F}_2$.

Hence $\sigma \in \mathcal{L}(\gamma, \mathcal{F}_1) \cap \mathcal{L}(\gamma, \mathcal{F}_2)$. The S_3 action on colours sends this σ to $|S_3| = 6$ distinct elements of the intersection (they're distinct because σ uses all three colours). $\square$

Proposition (Lower bound is achieved)**.** *The lower bound $|\cdots| \geq 6$ in Prop. is achieved exactly when G has a* unique *proper 3-edge-coloring up to S_3-relabelling — equivalently, when G is a single closed $C_{2k/3}$-cycle (with two alternating perfect matchings, plus a third perfect matching forced by the remaining edges). In particular, the worst case is $G \cong K_{3,3}$ (when $k = 9$) or G a single cycle, both of which arise empirically.*

(Sketch: a 3-regular bipartite graph has ≥ 6 proper 3-edge- colourings, with equality iff the colouring is essentially unique. The unique-up-to-S_3 case is exactly when G is highly symmetric. This matches the empirical observation that worst-case intersections are single S_3-orbits.)

What remains: the gap in the actual setting

In the real chain-pigeonhole setup, T_1's chord is on $B_{\text{in}}^{(1)} = \gamma$ (so T_1 *does* give a direct γ-face partition $\mathcal{F}_1$), but T_2's chord is on $B_{\text{in}}^{(2)}$, *not* on γ. So T_2 does not directly give a γ-face partition.

Conjectural extension to the real case

Conjecture (Induced γ-partition from T_2)**.** *For any SP tire T_2 with $B_{\text{in}}^{(2)}$-side chord structure $O^{(2)}$ such that every $O^{(2)}$-face has exactly 3 $B_{\text{in}}^{(2)}$-edges, the outer-spoke support $\pi_U(\mathcal{C}(T_2)) \subseteq \{1,2,3\}^\gamma$ contains a Latin subset $\mathcal{L}(\gamma, \widetilde{\mathcal{F}_2})$ for some induced face partition $\widetilde{\mathcal{F}_2}$ of γ into triples, where $\widetilde{\mathcal{F}_2}$ is determined by how T_2's annular triangulation distributes $B_{\text{in}}^{(2)}$ faces across γ-edges.*

If Conjecture holds, then Prop. immediately gives the worst-case overlap ≥ 6 for general adjacent tire pairs.

Why this is plausible

At $\gamma = 6$, $T_1 = (m_1, (0,3), \text{SP})$ vs $T_2 = (m_2, B_{\text{in}}^{(2)} = C_3, \text{SP})$, the empirical 6-element intersection consists of patterns (a,b,c,b,c,a) which:

- T_1's γ-face partition $\{0,1,2\} \mid \{3,4,5\}$: $\sigma|_{0,1,2} = (a,b,c)$ is a permutation, $\sigma|_{3,4,5} = (b,c,a)$ is a permutation. ✓
- The pattern uses each colour exactly twice (positions $\{0,5\},\{1,3\},\{2,4\}$). These pair into 3 groups of 2 — but not in a way that's directly a Latin γ-partition of size 3.

The T_2-side constraint forces a structural pattern (equal pairs at specific positions); the empirical fact is that T_2's induced constraint on γ *is* "something like a Latin partition," but I haven't yet found the precise statement.

One concrete attempt at the induced partition

For T_2 with $B_{\text{in}}^{(2)} = C_{k_2}$ and balanced annular triangulation, the dual cycle of T_2 alternates between D-triangles (one $B_{\text{in}}^{(2)}$-edge each) and U-triangles (one γ-edge each). Each $B_{\text{in}}^{(2)}$-edge has, in some sense, two "adjacent" γ-edges via the two annular edges of its D-triangle.

Candidate induced partition. Group the γ-edges by "which $B_{\text{in}}^{(2)}$-face's pair of γ-neighbours they share an annular edge with." When $k_2 = k/3$ (so there are $k/3$ D-triangles, each accounting for 3 γ-edges via its 2 annular edges), this gives a partition of $E(\gamma)$ into $k/3$ triples — the right shape.

Verifying that this candidate partition gives the correct Latin subset of π_U is an open computation.

Where the lower bound of $|S_3| = 6$ comes from independently

Setting aside the construction, the lower bound of 6 has a clean abstract origin:

Lemma (Lower bound from S_3-invariance)**.** *Both S_1 and S_2 are S_3-invariant (the S_3 action on colours acts on edge 3-colourings of G'). Hence $S_1 \cap S_2$ is S_3-invariant, so it decomposes into S_3-orbits. Any non-trivial S_3-orbit on $\{1,2,3\}^k$ where σ uses all three colours has size exactly $|S_3| = 6$. Constant σ (single colour) orbits have size 3, but constants are typically not in the support under*

SP chord constraints with ≥ 2 faces of size ≥ 2 (which forces σ to use both "other" colours on each face).

So the absolute floor is 6 once we rule out emptiness. The γ-partition / König argument is the cleanest way I know to rule out emptiness.

Summary of what's proven, conjectured, and open

- **Proven (Prop.):** When both T_1 and T_2 give direct all-3 γ-face partitions, $|S_1 \cap S_2| \geq 6$, witnessed by lifting a König 3-edge-colouring of the 3-regular bipartite "face-incidence" graph.
- **Lower-bound origin:** Both S_1 and S_2 are S_3-invariant, so any non-empty intersection has size at least 6 (the size of a non-constant S_3-orbit).
- **Conjectured (Conj.):** The real chain-pigeonhole setup, where T_2's chord is on $B_{\text{in}}^{(2)}$ rather than on γ, also reduces to Prop. via an induced γ-partition $\widetilde{\mathcal{F}_2}$. Empirical data is consistent with this conjecture but I haven't constructed $\widetilde{\mathcal{F}_2}$ explicitly.
- **Open:** Find the explicit map $T_2 \mapsto \widetilde{\mathcal{F}_2}$, prove the support inclusion, and combine with Prop. to close the worst-case lower bound.