The eight surviving γ-partitions at $k = 9$: a clean structural description

Setup

For $T_2 = (m_2 = 9, k_2 = 9, \text{chords} = \{(0,3),(3,6)\}, \text{SP})$, γ has length 9 and T_2's outerplanar $O^{(2)}$ has three 3-edge faces $F_A = \{0,1,2\}$, $F_B = \{3,4,5\}$, $F_C = \{6,7,8\}$ (indices into $B_{\text{in}}^{(2)}$). Empirically (`notes/induced_partition_findings.tex`): of the 280 triple-partitions of $\{0,\dots,8\}$, exactly 8 have $\mathcal{L} \subseteq \pi_U(T_2)$. We give a clean structural description.

Classifying γ-edges by T_2's face structure

In the balanced annular triangulation of T_2, D-positions on T'_{ann} are $\{0,2,4,6,8,10,12,14,16\}$. U-position $2i+1$ corresponds to γ-edge i. Each γ-edge is "between" two D-positions on T'_{ann}, and is classified by which $O^{(2)}$-faces those two D-positions belong to.

γ-edge i	U-position	adjacent D's (faces)	classification
0	1	$D{=}0, D{=}2$ (both F_A)	internal to F_A
1	3	$D{=}2, D{=}4$ (both F_A)	internal to F_A
2	5	$D{=}4(F_A), D{=}6(F_B)$	boundary F_A–F_B
3	7	$D{=}6, D{=}8$ (both F_B)	internal to F_B
4	9	$D{=}8, D{=}10$ (both F_B)	internal to F_B
5	11	$D{=}10(F_B), D{=}12(F_C)$	boundary F_B–F_C
6	13	$D{=}12, D{=}14$ (both F_C)	internal to F_C
7	15	$D{=}14, D{=}16$ (both F_C)	internal to F_C
8	17	$D{=}16(F_C), D{=}0(F_A)$ (cyclic)	boundary F_C–F_A

So γ partitions into 6 internal γ-edges (2 per face) and 3 boundary γ-edges (1 per pair of adjacent faces):

$$\text{Internal}_{F_A} = \{0,1\}, \quad \text{Internal}_{F_B} = \{3,4\}, \quad \text{Internal}_{F_C} = \{6,7\};$$

$$\delta_{AB} = 2, \quad \delta_{BC} = 5, \quad \delta_{CA} = 8.$$

The structural description

Proposition (Face-pair connection partitions). *A triple-partition $\widetilde{\mathcal{F}_2}$ of γ has $\mathcal{L}(\gamma, \widetilde{\mathcal{F}_2}) \subseteq \pi_U(T_2)$ iff it has the following structure: each block consists of*

- *one boundary γ-edge δ_{ij} between an adjacent $O^{(2)}$-face pair (F_i, F_j),*
- *one internal γ-edge from F_i (i.e. one of Internal_{F_i}'s two elements),*

- *one internal γ-edge from F_j.*

Equivalently: blocks are in bijection with adjacent $O^{(2)}$-face pairs (here, $\{AB, BC, CA\}$), and for each face F_i the two internal γ-edges are distributed between the two blocks "involving" F_i (one per block).

Proof of count. For $r = 3$ faces: 3 boundary edges (one per adjacent pair), so 3 blocks. Each face F_i has 2 internal γ-edges, and each internal must go to one of the two blocks involving F_i. Choices: 2 per face, r faces $\Rightarrow 2^r = 8$ partitions matching the empirical 8 survivors. $\square$

All 8 survivors enumerated with this structure

Writing each partition as $\{(\text{internal}_A, \delta_{AB}, \text{internal}_B),\ (\text{internal}'_B, \delta_{BC}, \text{internal}_C),\ (\text{internal}'_C, \delta_{CA}, \text{internal}'_A)\}$ where internal_A and $\text{internal}'_A$ split $\{0, 1\}$, etc.:

(F_A, F_B, F_C) split	resulting partition
$(0\|1, 3\|4, 6\|7)$	$\{(0,2,3),(4,5,6),(7,8,1)\}$
$(0\|1, 3\|4, 7\|6)$	$\{(0,2,3),(4,5,7),(6,8,1)\}$
$(0\|1, 4\|3, 6\|7)$	$\{(0,2,4),(3,5,6),(7,8,1)\}$
$(0\|1, 4\|3, 7\|6)$	$\{(0,2,4),(3,5,7),(6,8,1)\}$
$(1\|0, 3\|4, 6\|7)$	$\{(1,2,3),(4,5,6),(7,8,0)\}$
$(1\|0, 3\|4, 7\|6)$	$\{(1,2,3),(4,5,7),(6,8,0)\}$
$(1\|0, 4\|3, 6\|7)$	$\{(1,2,4),(3,5,6),(7,8,0)\}$
$(1\|0, 4\|3, 7\|6)$	$\{(1,2,4),(3,5,7),(6,8,0)\}$

These match the empirical survivors (up to relabeling block order).

Why the naive candidates fail

Both `induced_partition.py`'s candidate 1 (next-D) $\{(0,1,8),(2,3,4),(5,6,7)\}$ and candidate 2 (prev-D) $\{(0,1,2),(3,4,5),(6,7,8)\}$ *group both internal γ-edges of one face into one block.* E.g. candidate 2's first block $\{0,1,2\}$ contains both Internal_{F_A} edges (0 and 1) plus the boundary $\delta_{AB} = 2$ — no internal from F_B.

This violates the structural rule of Prop. (which requires one internal from *each* of the two adjacent faces). Empirically these partitions' Latin sets are not contained in $\pi_U(T_2)$.

Generalisation to r faces

For an SP tire whose $O^{(2)}$ has r all-3 faces arranged cyclically $F_0, F_1, \ldots, F_{r-1}$ on $B_{\text{in}}^{(2)}$, the γ-edge classification gives:

- $2r$ internal γ-edges (two per face),
- r boundary γ-edges (one per cyclically-adjacent pair (F_i, F_{i+1})),

for a total of $3r$ γ-edges, which is exactly $|\gamma| = k$ when $k = 3r$ (i.e. $k_2 = k$, the symmetric case).

Proposition (General structural description)**.** *For the symmetric case $k = k_2$ with r all-3 $O^{(2)}$-faces, the triple-partitions $\widetilde{\mathcal{F}_2}$ satisfying $\mathcal{L}(\gamma, \widetilde{\mathcal{F}_2}) \subseteq \pi_U(T_2)$ are exactly those of the form:*

- *Block $b_{i,i+1}$ for each adjacent pair, containing boundary $\delta_{i,i+1}$, one internal of F_i, one internal of F_{i+1},*
- *subject to: for each face F_i, its two internals are distributed between the two blocks $b_{i-1,i}$ and $b_{i,i+1}$ (one per block).*

Count: 2^r partitions.

Status. Proved (cleanly) at $k = 9$ by exhaustive verification matching the $8 = 2^3$ count. At $k = 6$ ($r = 2$), the proposition predicts $2^2 = 4$ "structural" partitions, which is a strict subset of the 10 triple-partitions whose Latin sets fit in $\pi_U(T_2)$; the extra 6 are absorbed by the relatively large $|\pi_U(T_2)| = 90$ at $k = 6$. At $k \geq 9$ we expect (and observe at $k = 9$) that the structural partitions are the only ones working.

Open: prove the proposition for all r

Prop. is currently empirical-only. A clean proof would presumably show:

1. *Necessity*: a Latin partition not of the face-pair- connection form contains a σ that violates a proper-edge-coloring constraint inside $T'_{f'}$.
2. *Sufficiency*: each of the 2^r structural partitions' Latin sets is realisable as π_U-projections, by an explicit construction lifting a structural assignment of colors at internal and boundary γ-edges into a proper coloring of T'_{ann} + spokes.

Implications for the König lift

The worst-case note's conjecture (*t2-induces-partition*) was that the induced γ-partition is unique (the next-D or prev-D candidate). The reality is that there are 2^r structurally valid candidates; the candidates from the worst-case note are *not* among them (they violate the "one internal per face per block" rule).

So the König-lift approach can be *rescued* by replacing the naive candidate $\widetilde{\mathcal{F}_2}$ with any of the 2^r face-pair-connection partitions, and applying the König argument on the bipartite face-incidence graph of $\mathcal{F}_1$ versus this new $\widetilde{\mathcal{F}_2}$. This is the natural next step in the program.