COLORING NESTED TIRE GRAPHS

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Abstract. We establish the foundational structure of nested level-induced tire decompositions of a plane triangulation G. A *level source* of G induces a BFS layering of G and endows the inner planar dual G' with a *dual depth* grading. The basic object of study is the *tire graph* T — a plane graph whose outer and inner boundaries bound a closed planar region, the *tire tread* R, triangulated by the *annular edges* E_{ann}. Our main structural result, the *tire-component lemma*, exhibits each connected component of G'_d as a tire graph; the *tire-tread partition theorem* consequence shows the resulting tire treads partition the bounded faces of G. Coloring questions on G thereby factor through coloring questions on the individual treads.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

The structural study of such a minimal counterexample is the overarching motivation for the present line of work. This first paper establishes the foundational vocabulary — level sources, dual depth, tire graphs, and partial tire duals — on which subsequent papers in the series build. In particular, the companion paper [3] uses these definitions to develop nested-cycle structure theorems and chain-pigeonhole conjectures for tire annular subgraphs of G'.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each

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vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \operatorname{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

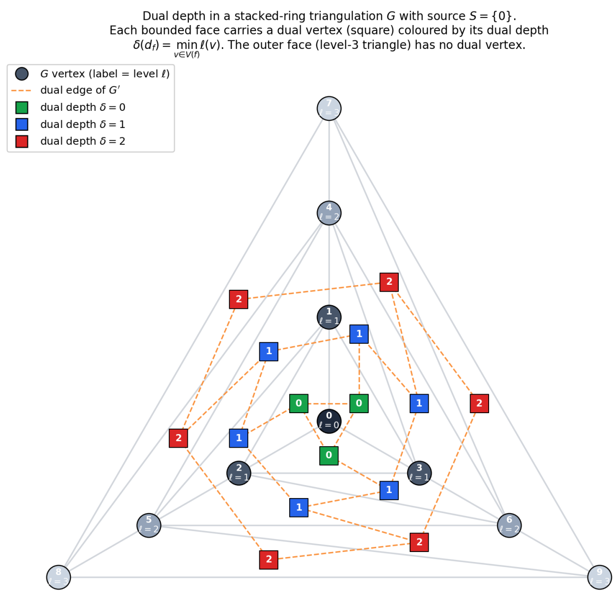

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.5 (Tire graph)**.** A *tire graph* consists of a plane graph T together with an *outer boundary* $B_{\text{out}} \subseteq T$ and an *inner outerplanar graph* $O \subseteq T$ with $V(B_{\text{out}}) \cap V(O) = \emptyset$, where

- B_{out} is either a simple cycle of length ≥ 3 or a single vertex (a *degenerate outer boundary*);
- O is an outerplanar graph; its *inner boundary* B_{in} is the closed walk in O that traces the boundary of O's outer face in the inherited embedding, which is a simple cycle when O is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if $|V(O)| = 1$, we say T has a *degenerate inner boundary*.

At most one of $B_{\text{out}}, B_{\text{in}}$ may be degenerate. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \qquad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T, the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R. We call R the *tire tread* of T.

When B_{out} is a simple cycle and O is 2-connected, the tread is a closed annulus. More generally, R is a closed planar region that may fail to be a 2-manifold at cut-vertices of O (where two "lobes" of the depth-d region meet at a single vertex); the inner boundary B_{in} is then a non-simple closed walk that visits the cut-vertex multiple times. The relaxed definition accommodates outerplanar inner graphs with bridges, cut-vertices, or multiple connected components. When either boundary is degenerate, the tread is a closed disk with that vertex as apex.

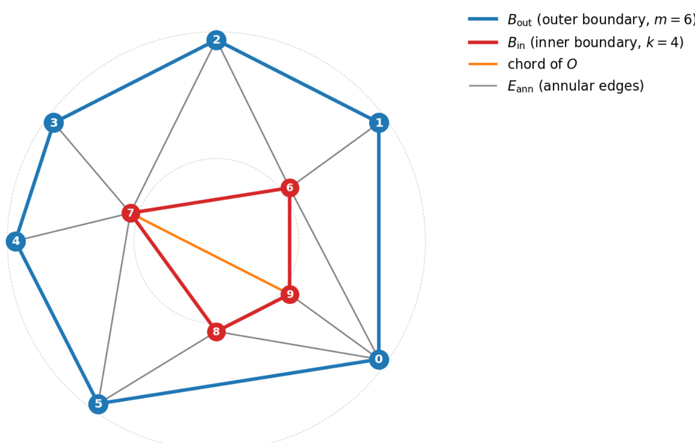

FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \ldots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \ldots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7\text{–}9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Remark 1.6. Let $m = |V(B_{\text{out}})|$ and $k = |V(B_{\text{in}})|$. By Euler's formula on the tire tread R, the tire graph has $m + k$ triangular faces inside R and $|E_{\text{ann}}| = m + k$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(m, k) = 1$), there are $m + k - 1$ triangular faces and $|E_{\text{ann}}| = m + k - 1$.

Proposition 1.7 (Source-side simple-cycle property)**.** *Let G be a maximal planar graph with planar embedding Π_G and single-vertex source v_0. Let $d \geq 1$, $v \in L_d$, and let C' be a connected component of G'_d such that v is incident to some face in $F_{C'}$. Then the depth-d faces in $F_{C'}$ incident to v form a single contiguous arc in v's rotation in Π_G.*

Equivalently: for any such component, the source-side boundary of $R_{C'}$ is a simple cycle in L_d (no cut-vertices at level d).

Proof. Suppose for contradiction that the depth-d faces in $F_{C'}$ at v lie in two or more disjoint arcs of v's rotation. Adjacent vertices in G differ in level by at most 1, so a face at v has depth exactly d iff both other vertices have level $\geq d$, and depth $\leq d-1$ iff at least one has level $d-1$. Hence the gaps between the depth-d arcs at v are populated by level-$(d-1)$ neighbours of v, occurring in at least two disjoint arcs of v's rotation. Pick p in one such gap and q in another.

The BFS ball $G[L_{<d}]$ is connected, so there exists a simple path P in $G[L_{<d}]$ from p to q. Define the closed walk

$$W := v \to p \to P \to q \to v.$$

Every vertex of P lies in $L_{<d}$, while $\ell(v) = d$, so v is distinct from every vertex of P; P is simple, so its internal vertices are distinct; and $p \neq q$ since they lie in different gaps. Hence W is a simple cycle in G.

By the Jordan curve theorem, the planar embedding of W divides Π_G into two regions. In v's rotation, the edges $v - p$ and $v - q$ lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by W. By choice of p, q, the two arcs of depth-d faces at v in $F_{C'}$ lie in different regions of W (i.e., one arc on each side).

Since C' is connected in G' and contains depth-d faces in both arcs, there is a dual path $f_1, f_2, \ldots, f_k$ in G'_d with $f_1, f_k \in F_{C'}$ incident to v in different arcs, and with the intermediate faces $f_2, \ldots, f_{k-1}$ not incident to v (a shortest such dual path). Consecutive faces f_i, f_{i+1} share an edge e_i of G; for $i \geq 2$, both endpoints of e_i lie in $L_{\geq d}$ (since neither f_i nor f_{i+1} is incident to v, all six vertices of these two triangles lie in $L_{\geq d}$). In particular, e_i shares no endpoint with W except possibly v — and v is excluded from $f_2, \ldots, f_{k-1}$.

A planar edge with neither endpoint on a simple closed planar curve W has both of its incident faces on the same side of W. Applying this to each e_i ($i \geq 2$) inductively: starting from f_2 on the same side of W as f_1 (their shared edge $e_1 = w - w'$ opposite to v in f_1 has $w, w' \in L_{\geq d}$ and hence is not on W), the path $f_2 \to f_3 \to \cdots \to f_{k-1} \to f_k$ stays on one side of W.

But f_1 and f_k lie on different sides of W (by construction), contradicting the conclusion that the entire path lies on one side. □

Lemma 1.8 (Tire-component lemma)**.** *Let G be a maximal planar graph and let $S \subseteq V(G)$ be a level source. Fix a plane embedding Π_G of G in which S lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For $d \geq 0$, let*

$$G'_d := G'\big[\{d_f \in V(G') : \delta_G(d_f) = d\}\big]$$

be the inner-dual subgraph on dual vertices of dual depth d, and let C' be a connected component of G'_d. Write $F_{C'} := \{f : d_f \in V(C')\}$, $V_{C'} := \bigcup_{f \in F_{C'}} V(f)$, and let $C := G[V_{C'}]$ inherit its embedding from Π_G. Set $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$.

Then C, with the inherited embedding, is a tire graph in the sense of Definition 1.5. Its outer boundary B_{out} is the side of $R_{C'}$ closer to S in Π_G, namely the level-d subgraph $G[V_{C'} \cap L_d]$ (a simple cycle or single vertex); its inner outerplanar graph is $O = G[V_{C'} \cap L_{d+1}]$, and its inner boundary B_{in} is the outer-face boundary closed walk of O in the inherited embedding (a simple cycle when O is 2-connected, *a non-simple closed walk in general). The triangular faces of C inside the closed boundary region are exactly the faces of G in $F_{C'}$.*

Proof. Outerplanarity of the two level parts. By construction S lies on the outer face of Π_G, so the outerplanarity lemma of [2] applies directly with (G, Π_G, S), giving that $G[L_{d'}]$ is outerplanar for each $d' \geq 0$. Subgraphs of outerplanar graphs are outerplanar, so $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are both outerplanar.

Layer containment. Each $f \in F_{C'}$ has at least one vertex at level d, and adjacent vertices in G differ in level by at most 1; combined with $\delta_G(d_f) = d$, this forces $V(f) \subseteq L_d \cup L_{d+1}$. Hence $V_{C'} \subseteq L_d \cup L_{d+1}$, and C has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$.

Boundary edges are monochromatic in level. Each edge e on $\partial R_{C'}$ separates a face $f \in F_{C'}$ from a face $f' \notin F_{C'}$. Because f and f' share the edge e, their dual vertices are adjacent in G'; if both had depth d they would lie in the same component of G'_d, contradicting $d_f \in C'$ and $d_{f'} \notin C'$. Hence $\delta_G(d_{f'}) \neq d$; combined with the bounded-step property of δ across G'-adjacent faces, $\delta_G(d_{f'}) \in \{d-1, d+1\}$.

- If $\delta_G(d_{f'}) = d - 1$, the third vertex w of $f' = \{u, v, w\}$ (where u, v are the endpoints of e) has $\ell(w) = d - 1$. Each of u, v has $\ell \in \{d, d+1\}$ (from $V(f) \subseteq L_d \cup L_{d+1}$) and is adjacent to w, forcing $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$.
- If $\delta_G(d_{f'}) = d+1$, then all three vertices of f' lie in $L_{\geq d+1}$, so in particular $\ell(u) = \ell(v) = d + 1$.

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of $\partial R_{C'}$ is monochromatic in level.

Boundary structure. Each connected component of $\partial R_{C'}$ traces a closed walk in G that, by the monochromaticity above, lies entirely in L_d or entirely in L_{d+1}. By Proposition 1.7, the depth-d faces of $F_{C'}$ at any $v \in L_d \cap V_{C'}$ form a single contiguous arc in v's rotation, so the source-side boundary walk visits each L_d-vertex of $V_{C'}$ exactly once: it is a simple cycle. At vertices $v \in L_{d+1} \cap V_{C'}$ the depth-d faces may split into multiple arcs of v's rotation; this corresponds exactly to v being a cut-vertex of O, and the inner-side boundary walk visits v correspondingly many times — which is already accommodated by Definition 1.5 (where B_{in} is the outer-face boundary closed walk of O, not necessarily a simple cycle).

Outer boundary. Because S lies on the outer face of Π_G, the boundary curve(s) of $R_{C'}$ on the L_d side are closer to S in the embedding. In the inherited embedding of C, the unique unbounded face is the merged region containing the rest of Π_G outside $R_{C'}$ on the S side, so its boundary — a simple cycle on L_d (or a single vertex when $V_{C'} \cap L_d = \{v_0\}$, the $d = 0$ case) — serves as B_{out}. We set $B_{\text{out}} := G[V_{C'} \cap L_d]$ if this is a cycle, and the single vertex $\{v_0\}$ in the degenerate case.

Inner outerplanar graph. By the outerplanarity lemma of [2], $G[V_{C'} \cap L_{d+1}]$ is outerplanar. We set $O := G[V_{C'} \cap L_{d+1}]$. The boundary curve(s) of $R_{C'}$ on the L_{d+1} side are exactly the boundary of O's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around O from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary B_{in}. No further restriction on O's internal structure is needed: when $R_{C'}$ has more than two boundary components in the surface-classification sense (i.e. several disjoint simple cycles on L_{d+1}), these correspond precisely to the multiple connected components or bridge crossings of O, and the outer-face boundary closed walk of O captures them collectively.

Tire structure. The triangular faces of C inside the closed boundary region are by construction the depth-d faces in $F_{C'}$, and the edges of C are $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$ where E_{ann} are the edges of G between $V_{C'} \cap L_d$ and $V_{C'} \cap L_{d+1}$ that bound a face of $F_{C'}$. $\square$

Theorem 1.9 (Tire treads partition the bounded faces)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a level source lying on the outer face. For each $d \geq 0$ and each connected component C' of G'_d, let $T^{(d,C')}$ denote the tire graph supplied by Lemma 1.8, with tire tread $R_{C'} \subseteq |\Pi_G|$. Then the collection of treads*

$$\mathcal{R}(G,S) \ := \ \big\{ R_{C'} \ : \ d \geq 0,\ C' \ \textit{a connected component of}\ G'_d \big\}$$

partitions the bounded part of $|\Pi_G|$:

(i) *every bounded face f of G is contained in exactly one tread $R_{C'} \in \mathcal{R}(G,S)$;*
(ii) *distinct treads in $\mathcal{R}(G,S)$ have disjoint interiors and may share only boundary edges or vertices.*

Proof. Existence and uniqueness. Each bounded face $f \in F(G)$ has a uniquely-defined dual depth $\delta_G(d_f) \in \mathbb{Z}_{\geq 0}$, so the dual vertex d_f lies in G'_d for $d = \delta_G(d_f)$ and in no other $G'_{d'}$. Within G'_d, the vertex d_f belongs to exactly one connected component C'. By Lemma 1.8, $F_{C'}$ is precisely the set of faces $f' \in F(G)$ with $d_{f'} \in V(C')$; in particular $f \in F_{C'}$, hence $f \subseteq R_{C'}$.

For any other tread $R_{C''} \in \mathcal{R}(G,S)$, the component C'' is either at a different depth $d' \neq d$ (in which case $F_{C''}$ consists of depth-d' faces and $f \notin F_{C''}$) or at depth d but a different component $C'' \neq C'$ (in which case the two components are vertex-disjoint in G'_d, so again $f \notin F_{C''}$). In both cases $f \notin R_{C''}$ (more precisely, f is not one of the triangular faces of G in $F_{C''}$, so f's interior is not contained in $R_{C''}$).

Disjoint interiors. Each tread $R_{C'}$ is the union of its triangular faces $F_{C'} \subseteq F(G)$; distinct treads correspond to disjoint $F_{C'}$ (by the argument above), and the interiors of distinct G-faces are disjoint. Hence interiors of distinct treads are disjoint.

Coverage. Conversely, every bounded $f \in F(G)$ has $d_f \in V(G')$ with some dual depth d, and thus lies in $R_{C'}$ where C' is its component of G'_d. So $\bigcup_{R \in \mathcal{R}(G,S)} R$ contains every bounded face of G. $\square$

Remark 1.10. Either boundary part of C in Lemma 1.8 may be degenerate. At $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ as the degenerate *outer* boundary and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G) as the inner boundary. Symmetrically, at $d = D_{\max}$, $V_{C'} \cap L_{D_{\max}+1} = \emptyset$ degenerates to a single deepest vertex serving as the *inner* boundary, with the level-$D_{\max}$ cycle as the outer boundary.

Remark 1.11. Two structural features of $R_{C'}$ that might at first appear to obstruct the tire-graph conclusion are both already accommodated by Definition 1.5:

Cut-vertices of O. A vertex $v \in V_{C'} \cap L_{d+1}$ may have the faces of $F_{C'}$ incident to it split into two or more arcs in v's rotation in Π_G, separated by faces of higher depth. This corresponds exactly to v being a cut-vertex of $O = G[V_{C'} \cap L_{d+1}]$, and the inner boundary closed walk B_{in} then visits v multiple times — once for each arc. No additional hypothesis is needed.

Multi-hole topology of $R_{C'}$. Even when $R_{C'}$ encloses several disjoint depth-$> d$ sub-regions, the inner outerplanar graph O captures the multi-hole structure as a disconnected or non-2-connected outerplanar graph (with bridges or multiple components), and its outer-face boundary closed walk serves as B_{in} traversing bridges twice and visiting cut-vertices multiple times.

In the special case $d = 0$ with single-vertex source $S = \{v_0\}$, $R_{C'}$ is the star of v_0, a topological closed disk with one boundary cycle (the link of v_0); the corresponding tire graph has degenerate outer boundary $\{v_0\}$.

Theorem 1.12 (Inner dual of a tire tread is outerplanar)**.** *Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph, and let Γ be the graph on vertex set $\{d_f : f \in F_{\text{ann}}\}$ with an edge $d_f d_{f'}$ for each interior annular edge of T (= each edge of T whose two incident faces both lie in F_{ann}). Then Γ is outerplanar.*

Moreover, Γ admits a planar embedding as a (possibly non-simple) Hamilton walk through every d_f, plus zero or more non-crossing chords.

Proof. We argue by cases on whether the tire tread R is a disk or an annulus.

Case 1: R is a closed disk (one of $B_{\text{out}}, B_{\text{in}}$ degenerate, by Definition 1.5). Let v_0 be the degenerate-boundary vertex (the apex) and let $k = |B_{\text{non-deg}}|$ be the length of the non-degenerate boundary cycle. The triangulation of R is a *fan* of k triangles around v_0: each triangle has the form $\{v_0, u_i, u_{i+1}\}$ where $u_1, \dots, u_k$ are the boundary-cycle vertices in cyclic order. Each triangle has two spoke edges (= the two edges incident to v_0, shared with the two neighbouring fan triangles) and one boundary edge (in $B_{\text{non-deg}}$, contributing a leaf in $D(T)$ but no edge in Γ). Hence every d_f has Γ-degree exactly 2, and Γ is a single cycle of length k. Cycles are outerplanar.

See Figure 3 for the disk case ($k = 6$).

Case 2: R is an annulus (both B_{out} and B_{in} non-degenerate). We construct an explicit outerplanar embedding of Γ as a Hamilton walk plus non-crossing chords.

Step 1: Cyclic ordering of F_{ann}. The boundary of the annular tread is the disjoint union $\partial R = B_{\text{out}} \sqcup \overline{B_{\text{in}}}$ (viewing B_{in} as a closed walk traced in the appropriate orientation). Each boundary edge of R is incident to exactly one annular face: walking around B_{out} in cyclic order produces a sequence $f_1^{\text{out}}, f_2^{\text{out}}, \dots, f_n^{\text{out}}$ of (not necessarily distinct) annular faces, one per B_{out}-edge; similarly walking around B_{in} produces a sequence $f_1^{\text{in}}, \dots, f_{m_\partial}^{\text{in}}$ where m_∂ is the length of the inner-boundary walk. Pick any spoke $e^\star = uw \in E_{\text{ann}}$ with $u \in V(B_{\text{out}})$ and $w \in V(B_{\text{in}})$; cut R along $e^\star$. This converts the annulus into a closed disk $\tilde{R}$ whose boundary walks once around B_{out}, once along $e^\star$, once around B_{in} in reverse, and once back along $e^\star$. Concatenating the two boundary sequences (in the order dictated by this disk traversal) yields a single cyclic sequence

$$\mathcal{S} = (f_1^{\text{out}}, \dots, f_n^{\text{out}}, f_1^{\text{in}}, \dots, f_{m_\partial}^{\text{in}})$$

of annular faces with multiplicities.

Step 2: The Hamilton walk. Consecutive entries of $\mathcal{S}$ correspond either to the same annular face (when two adjacent boundary edges meet at a vertex incident to a single annular face) or to two annular faces sharing an interior edge of E_{ann}. In the

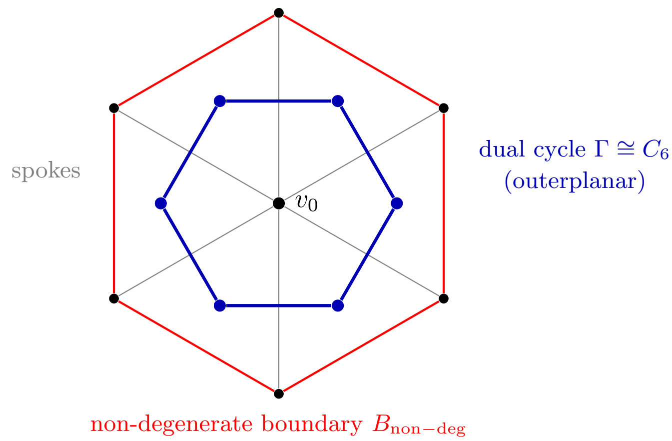

FIGURE 3. Case 1 (R = disk, $k = 6$). The apex v_0 sits at the centre; the non-degenerate boundary $B_{\mathrm{non-deg}}$ (red) is the hexagonal outer cycle; spokes (grey) triangulate the disk into a fan of 6 triangles around v_0. Each triangle has two spoke edges (interior, contributing Γ-edges) and one boundary edge (contributing a leaf in $D(T)$, no Γ-edge). The inner dual Γ (blue) is the cycle C_6 formed by the six annular face centroids, a manifestly outerplanar graph.

former case the walk stays at one Γ-vertex; in the latter it uses one Γ-edge. The resulting closed walk in Γ visits every face that appears in $\mathcal{S}$ at least once.

If every $f \in F_{\mathrm{ann}}$ appears in $\mathcal{S}$ (i.e. every annular face has at least one boundary edge of R), the walk is a Hamilton walk in Γ, and we are done up to Step 3. Each annular face with two boundary edges contributes a vertex visited twice; each with three contributes a vertex visited three times.

If some $f \in F_{\mathrm{ann}}$ does not appear in $\mathcal{S}$ (i.e. has no boundary edge of R), then all three edges of f are interior annular edges, so d_f has degree 3 in Γ. Such a face is "trapped" in the interior of the dual graph and appears as the endpoint of a chord. Extend the walk by: whenever it crosses an interior annular edge e shared with a boundary-free face f, detour through f and back. After finitely many such detours (one per boundary-free face), the walk becomes a Hamilton walk visiting every d_f.

Step 3: Non-crossing chords. The Γ-edges not used by the Hamilton walk constructed in Step 2 are the remaining interior annular edges. Each such edge $e \in E_{\mathrm{ann}}$ corresponds to a chord between two non-adjacent positions of $\mathcal{S}$. In the inherited planar embedding of Γ in R, these chords are drawn as straight segments between annular triangle centroids; *they do not cross* because the underlying E_{ann} edges they cross are themselves non-crossing in the planar embedding of T.

Step 4: Outerplanar embedding. We now lay out Γ as follows: place the $|F_{\mathrm{ann}}|$ vertices on a circle in the cyclic order given by $\mathcal{S}$ (treating multiply-visited faces as single circle vertices). Connect consecutive vertices on the circle by the Hamilton-walk edges, which forms the closed walk. Draw the remaining edges as chords inside the circle. Because the chords were non-crossing in T's planar embedding, they remain non-crossing here. All vertices lie on the outer face (the unbounded region outside the circle), making Γ outerplanar. □ □

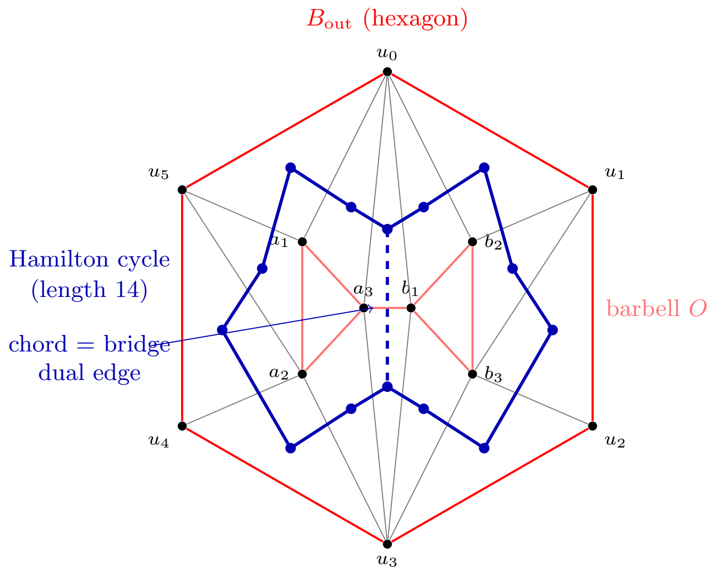

FIGURE 4. Case 2 (R = annulus) with O a barbell. B_{out} is the outer hexagon (red); O has two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ joined by the bridge a_3–b_1 (all light red). The annulus is triangulated by 14 annular triangles: 6 "outer-cap" triangles (one per outer edge), 6 "inner-cap" triangles (one per non-bridge edge of O), and 2 "bridge-cap" triangles $\{u_0, a_3, b_1\}$ and $\{u_3, a_3, b_1\}$ adjacent to the bridge. Each blue dot sits at the centroid of an annular triangle; blue edges connect dual vertices whose triangles share an interior annular edge (spoke or bridge). The two bridge-cap vertices have Γ-degree 3 (their triangles have no boundary edge) and are joined by the dashed blue *chord* corresponding to the bridge; the remaining 13 edges form the Hamilton cycle that wraps around the annulus. All 14 vertices lie on the outer face of the cycle-with-chord embedding, so $\Gamma \cong \Theta(1, 7, 7)$ is outerplanar.

Remark 1.13. In the *spoke-only* case (Definition 1.5 with O 2-connected and E_{ann} consisting only of spokes), every annular face has exactly one boundary edge, every d_f has Γ-degree 2, and the construction of the Theorem 1.12 proof reduces to the classical Hamilton cycle $\Gamma \cong C_{n+m}$ with zero chords.

Remark 1.14. When O has a bridge $e_{\text{br}} \in E(O)$ whose two incident faces are annular triangles, e_{br} contributes an interior annular edge in Γ rather than two leaves in $D(T)$ (see Definition 1.7 of [3]). The two bridge-incident annular triangles have Γ-degree 3; the resulting Γ has the structure of a Hamilton cycle of length $n + m_\partial$ plus a single chord (length 1). This corresponds to the theta graph $\Theta(1, b, c)$ identified empirically in [3], which has no $K_{2,3}$ subdivision (since one of the three paths has length 1 and so contributes no degree-2 branch vertex), hence is outerplanar as predicted.

Theorem 1.15 (Tait correspondence: 4-colorings of a tire vs 3-edge-colorings of its inner dual)**.** *Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph (viewed as an annular triangulation of its tire tread R) and let Γ be its inner dual (Theorem 1.12). Then*

$$\#\{\textit{proper } 4\textit{-vertex-colorings of } T\}/|S_4| \;=\; \#\{\textit{proper } 3\textit{-edge-colorings of } \Gamma\}/|S_3|.$$

That is, the number of 4-vertex-colorings of T up to permutation of the colour set $\{0,1,2,3\}$ equals the number of 3-edge-colorings of Γ up to permutation of the colour set $\{1,2,3\}$.

Proof. The argument is the classical Tait correspondence [1] adapted to the annular triangulation T. Encode the four colours of a proper 4-vertex-coloring $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. For each interior annular edge e of T (whose two incident faces both lie in F_{ann}, contributing a Γ-edge e^*), set

$$\chi^*(e^*) \;:=\; c(u) + c(v) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \text{where } u, v \text{ are the endpoints of } e.$$

Since $c(u) \neq c(v)$, we have $\chi^*(e^*) \neq 00$, so χ^* takes values in $\{01, 10, 11\}$, which we identify with the 3-edge-coloring palette $\{1, 2, 3\}$.

Properness. At each Γ-vertex d_f corresponding to an annular triangle $f = \{u, v, w\}$, the three incident Γ-edges (one per cycle-edge of f) carry colours $c(u) + c(v),\ c(v) + c(w),\ c(u) + c(w)$. These three elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ sum to 0 and are pairwise distinct (their pairwise differences are $c(u) - c(w),\ c(v) - c(u),\ c(w) - c(v)$, each nonzero), so they form a permutation of $\{01, 10, 11\}$ — a proper edge colouring at d_f.

Surjectivity onto cosets. Given a proper 3-edge-coloring χ^* of Γ, the equation $c(u) + c(v) = \chi^*(e^*)$ admits exactly $|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$ solutions $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ (a global translation is the only freedom). Hence the map $c \mapsto \chi^*$ is 4-to-1.

Count. Therefore $\#\{\text{4-colorings of } T\} = 4 \cdot \#\{\text{3-edge-colorings of } \Gamma\}$. Dividing by $|S_4| = 24$ on the left and $|S_3| = 6$ on the right (since S_4 acts faithfully on the 4-colorings and S_3 on the 3-edge-colorings, and the 4-to-1 map respects the $S_4/S_3 \cong S_3$ quotient via the natural surjection $S_4 \twoheadrightarrow S_3$) gives the stated equality. ☐

Theorem 1.16 (Count formula for spoke-only and single-chord tires)**.** *Let T be a tire graph with $|F_{\text{ann}}| = n$ annular triangles, and let Γ be its inner dual.*

(i) *If $\Gamma \cong C_n$ (the spoke-only case of Remark 1.13), then*

$$\#\{\textit{proper } 3\textit{-edge-colorings of } \Gamma\} \;=\; 2^n + 2 \cdot (-1)^n.$$

(ii) *If $\Gamma \cong \Theta(1, b, c)$ with $b + c = n$ (the single-chord case of Remark 1.14), then*

$$\#\{\textit{proper } 3\textit{-edge-colorings of } \Gamma\} \;=\; 6\,(\alpha_b\alpha_c + \beta_b\beta_c),$$

where $\alpha_L = \big(2^{L-1} + 2(-1)^{L-1}\big)/3$ and $\beta_L = \big(2^{L-1} - (-1)^{L-1}\big)/3$.

Proof. (i) Standard chromatic polynomial of the cycle: $P(C_n, k) = (k-1)^n + (-1)^n(k-1)$. At $k = 3$: $2^n + 2(-1)^n$ (cf. Proposition 1.2 of [3]).

(ii) By transfer matrix on the two non-chord paths of $\Theta(1, b, c)$ with the chord edge colour fixed and the two endpoint colour assignments enumerated. At each trivalent endpoint, the two path-incident edges receive the two non-chord colours (2 ways). Conditional on these assignments, the path's interior edges form a 3-edge-coloring of a path of length b (resp. c) with both endpoints' colours fixed; the number

of such colorings is the (c_a, c_b)-entry of T^{L-1}, where $T = J - I$ is the 3×3 adjacency matrix of the colour-difference graph. T^{L-1} has diagonal entries α_L and off-diagonal entries β_L. Summing over the four endpoint configurations ($\{x, y\}, \{x', y'\} = \{2, 3\}$ each in two orderings) and multiplying by the three chord colour choices gives the stated formula. Verification: $\Theta(1, 2, 2)$ ($= K_4 \backslash e$, $\alpha_2 = 0$, $\beta_2 = 1$) yields $6\cdot(0+1) = 6$ proper 3-edge-colorings, matching the known count for $K_4 \setminus e$. $\square$

Remark 1.17. For an inner dual Γ with more than one non-crossing chord, the count depends on the chord structure, not just on the pair (number of vertices, number of chords). Two outerplanar graphs with the same n and k can have different proper 3-edge-coloring counts depending on how the chords are arranged (nested, sequential, sharing vertices, etc.). However, every such count can be computed in linear time by tree-decomposition methods, since outerplanar graphs have treewidth at most 2 and the edge-chromatic polynomial admits a deletion–contraction recursion that respects the cycle-plus-chord structure.

Theorem 1.18 (Tire treads form a rooted tree under face containment)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a single-vertex level source $\{v_0\}$ lying on the outer face of Π_G. The collection $\mathcal{R}(G, S)$ of tire treads (Theorem 1.9) carries a canonical rooted tree structure $\mathcal{T}(G, S)$ defined as follows.*

- Root. *The depth-0 tire tread T_0 — the unique tire produced by Lemma 1.8 at $d = 0$, with degenerate outer boundary $B_{\text{out}} = \{v_0\}$ and inner outerplanar graph $O^{(0)} = G[L_1]$ — is the root.*
- Parent. *For each tire tread T_c at depth $d \geq 1$, its outer boundary $B^{(c)}_{\text{out}}$ is a cycle in L_d. The* parent *of T_c is the unique tire tread T_p at depth $d - 1$ whose inner outerplanar graph $O^{(p)}$ has $B^{(c)}_{\text{out}}$ as the boundary cycle of one of its bounded faces. Equivalently, R_c lies inside this bounded face of $O^{(p)}$ (which is itself the region of the plane cut off by $B^{(c)}_{\text{out}}$ on the side away from S).*
- Children. *The children of a tire tread T_p are in bijection with those bounded faces of $O^{(p)}$ whose interiors contain at least one vertex of G at level $\geq d+2$ — equivalently, with the connected components of G'_{d+1} whose tires have outer boundary cycle equal to a bounded face of $O^{(p)}$.*

Every tire tread except T_0 has exactly one parent; a tire tread may have zero, one, or several children.

Proof. Root is well-defined. At $d = 0$ with single-vertex source $S = \{v_0\}$, the dual subgraph G'_0 is connected (every face of G incident to v_0 has dual depth 0, and they form a single fan around v_0). By Lemma 1.8, the unique component of G'_0 gives the depth-0 tire T_0 described above.

Existence of parent. Fix a tire tread T_c at depth $d \geq 1$ arising from a connected component C'_c of G'_d. Its outer boundary $B^{(c)}_{\text{out}} = G[V_{C'_c} \cap L_d]$ is a simple cycle in L_d (Lemma 1.8; the source-side boundary of a tire is always a simple cycle, by Proposition 1.7). The faces of G immediately outside $B^{(c)}_{\text{out}}$ on the side facing S have depth $d-1$ (one of their three vertices lies in L_{d-1}, two in L_d). Let C'_p be the connected component of G'_{d-1} containing the dual vertex of any such face.

Uniqueness of parent. $B^{(c)}_{\text{out}}$ is a single simple cycle in G, with a well-defined "S-side" (the side of the cycle closer to v_0 in Π_G). The depth-$(d-1)$ faces lying on this side form a single contiguous arc around $B^{(c)}_{\text{out}}$ in the dual — they are all G'-adjacent in sequence (each pair of consecutive arc faces shares an edge in $B^{(c)}_{\text{out}}$). Hence they all lie in the same connected component C'_p of G'_{d-1}, which is therefore unique.

$B^{(c)}_{\text{out}}$ bounds a face of $O^{(p)}$. The parent tire T_p has $V(O^{(p)}) = V_{C'_p} \cap L_d \supseteq V(B^{(c)}_{\text{out}})$. The cycle $B^{(c)}_{\text{out}}$ is a subgraph of $O^{(p)}$ that bounds a face of $O^{(p)}$ in the inherited embedding: the cycle traces around a depth-$\geq d+1$ region (containing R_c and any descendants of T_c), which is exactly a bounded face of $O^{(p)}$.

Children description. The bounded faces of $O^{(p)}$ are in bijection with the connected components of G'_d whose faces lie inside those bounded regions (= one component per bounded face, by an argument analogous to the existence-and- uniqueness step above, applied one level deeper).

Tree property. Every non-root T_c has a unique parent at strictly smaller depth. Iterating the parent map strictly decreases depth, terminating at T_0. No cycles can form (depth is monotone). Hence $\mathcal{T}(G, S)$ is a rooted tree. □

Remark 1.19. A parent tire T_p has multiple children precisely when its inner outerplanar graph $O^{(p)}$ has multiple bounded faces with non-trivial interiors (= containing depth-$\geq d+2$ vertices of G). This happens, for instance, when $O^{(p)}$ has chords or cut-vertices that subdivide its inner region, or when $O^{(p)}$ has multiple connected components in $G[L_{d+1}] \cap V_{C'_p}$. By contrast, if $O^{(p)}$ is a simple cycle (the spoke-only case of Remark 1.13) with a non-empty interior, T_p has exactly one child.

Remark 1.20. Combining Theorem 1.9 (treads partition the bounded faces of G) with Theorem 1.18 (treads form a rooted tree), any proper coloring problem on G's bounded faces factors through:

- local coloring problems on each tread (the inner dual of each tread is outerplanar by Theorem 1.12), plus
- consistency constraints along parent-child interfaces (the cycle $B^{(c)}_{\text{out}}$ shared between a child and the face of its parent's $O^{(p)}$).

This is the structural setup underlying the chain-pigeonhole program for tire treads.

References

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