HUMANS SUFFICE: A NOVEL PROOF OF THE FOUR COLOR THEOREM

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Abstract.

1. Kempe's Proof (Valid Portion)

1.1. **Setup.** Kempe's strategy, published in 1879, follows a *minimal counterexample* argument. Suppose, for contradiction, that there exists a planar graph requiring 5 colors. Among all such graphs, let G be one with the fewest vertices. Then every planar graph with fewer vertices than G is 4-colorable, but G itself is not.

1.2. **Every Planar Graph Has a Vertex of Degree at Most 5.**

Lemma 1.1. *Every planar graph has at least one vertex of degree* ≤ 5.

Proof. Let G be a connected planar graph with V vertices, E edges, and F faces. By Euler's formula,

$$V - E + F = 2.$$

Since every face is bounded by at least 3 edges and each edge borders at most 2 faces, we have $2E \geq 3F$, hence $F \leq \frac{2E}{3}$. Substituting into Euler's formula:

$$V - E + \frac{2E}{3} \geq 2 \implies E \leq 3V - 6.$$

If every vertex had degree ≥ 6, then $2E \geq 6V$, so $E \geq 3V$, contradicting $E \leq 3V - 6$. Therefore at least one vertex has degree ≤ 5. $\square$

Since G is a minimal counterexample, it must contain a vertex v of degree at most 5. Kempe argued by cases on the degree of v.

1.3. **Cases of Degree at Most 3.** Suppose v has degree ≤ 3. Remove v from G to obtain the graph $G - v$. Since $G - v$ has fewer vertices than G, it is 4-colorable by minimality. Fix such a 4-coloring. Now reinsert v: its at most 3 neighbors occupy at most 3 of the 4 colors, so at least one color remains available for v. This yields a valid 4-coloring of G, a contradiction.

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1.4. **Case of Degree 4.** Suppose v has degree exactly 4 with neighbors a, b, c, d appearing in cyclic order around v in the planar embedding. Remove v and 4-color $G - v$ by minimality. If the four neighbors do not all receive distinct colors, then at least one color is unused among them, and we may assign that color to v, giving a contradiction.

So assume a, b, c, d receive all four distinct colors; call them 1, 2, 3, 4 respectively. Define a *Kempe chain* to be a maximal connected subgraph whose vertices are colored with exactly two specified colors.

Consider the Kempe chain K_{13} containing a (using colors 1 and 3).

- **Case 1**: c is not in K_{13}. Swap colors 1 and 3 throughout K_{13}. This is still a valid coloring of $G - v$, and now a receives color 3, so color 1 is free for v.
- **Case 2**: c is in K_{13}. Then there is a path of alternating colors 1 and 3 from a to c in the planar embedding. Because a and c alternate around v with b and d, this path separates b from d in the plane. Therefore b and d lie in different Kempe chains for colors 2 and 4. Swap colors 2 and 4 in the chain containing b; now b receives color 4, freeing color 2 for v.

In both cases we obtain a valid 4-coloring of G, a contradiction.

2. Resolution of Degree 5 Case

2.1. **At Least 12 Vertices of Degree 5.** Since we have already handled every vertex of degree ≤ 4, we may assume without loss of generality that the minimal counterexample G has minimum degree 5. We now show that G must contain at least 12 vertices of degree exactly 5.

Lemma 2.1. *If G is a planar graph with minimum degree 5, then G contains at least 12 vertices of degree exactly 5.*

Proof. For each $k \geq 5$, let n_k denote the number of vertices of degree exactly k in G. Since every vertex has degree ≥ 5,

$$V = \sum_{k \geq 5} n_k, \qquad 2E = \sum_{k \geq 5} k\, n_k.$$

Using $E \leq 3V - 6$, we obtain $2E \leq 6V - 12$, so

$$\sum_{k \geq 5} k\, n_k \leq 6 \sum_{k \geq 5} n_k - 12.$$

Moving all terms to the right-hand side gives

$$12 \leq 6 \sum_{k \geq 5} n_k - \sum_{k \geq 5} k\, n_k = \sum_{k \geq 5} (6 - k)\, n_k.$$

We now expand this sum by separating the $k = 5$, $k = 6$, and $k \geq 7$ terms:

$$\sum_{k \geq 5} (6 - k)\, n_k = \underbrace{(6 - 5)\, n_5}_{= n_5} + \underbrace{(6 - 6)\, n_6}_{= 0} + \sum_{k \geq 7} (6 - k)\, n_k = n_5 - \sum_{k \geq 7} (k - 6)\, n_k.$$

Therefore $12 \leq n_5 - \sum_{k \geq 7} (k - 6)\, n_k$, which rearranges to

$$n_5 \geq 12 + \sum_{k \geq 7} (k - 6)\, n_k.$$

Since $k - 6 \geq 1 > 0$ and $n_k \geq 0$ for all $k \geq 7$, each term in the sum is non-negative, so $\sum_{k \geq 7} (k - 6)\, n_k \geq 0$ and thus $n_5 \geq 12$. $\square$

2.2. **Two Non-Adjacent Vertices of Degree 5.**

Lemma 2.2. *If G is a planar graph with minimum degree 5, then G contains two non-adjacent vertices of degree exactly 5.*

Proof. By the previous lemma, G contains at least 12 vertices of degree exactly 5. Let S denote the set of all degree-5 vertices, so $|S| \geq 12$. Suppose for contradiction that every two vertices in S are adjacent, i.e., S induces a clique in G. Then every vertex $v \in S$ is adjacent to all other $|S| - 1 \geq 11$ vertices of S. But $\deg(v) = 5$, so v has at most 5 neighbors in total. Since $11 > 5$, this is a contradiction. Therefore S does not form a clique, and there exist two vertices $v_1, v_2 \in S$ that are non-adjacent. □

2.3. **The Reduced Subgraph G'.** By the previous lemma, we may fix two non-adjacent vertices $v_0, v_1 \in G$ each of degree 5. Define

$$G' = G - \{v_0, v_1\},$$

as the subgraph of G obtained by deleting v_0 and v_1 together with all edges incident to either vertex. Since G' has strictly fewer vertices than G, the minimality of G guarantees that G' admits a proper 4-coloring $\phi : V(G') \to \{1, 2, 3, 4\}$. Because v_0 and v_1 are non-adjacent in G, neither is a neighbor of the other, so each retains all 5 of its neighbors in G'. Let $N(v_i)$ denote the set of neighbors of v_i in G; then $N(v_0)$ and $N(v_1)$ are each sets of 5 vertices, all of which lie in G' and are assigned colors by ϕ. Note also that $N(v_0)$ and $N(v_1)$ may overlap. Let Φ be the set of all possible 4-colorings of G'.

2.4. **Not All Colorings in Φ Saturate Both Neighborhoods.**

Lemma 2.3. *It is not possible that every $\phi \in \Phi$ satisfies both $|\phi(N(v_0))| = 4$ and $|\phi(N(v_1))| = 4$.*

Proof. We show the stronger fact that there exists $\phi \in \Phi$ with $|\phi(N(v_0))| \leq 3$.

Since v_1 is a single vertex, $G - \{v_1\}$ is a planar graph on $V - 1$ vertices. Because $V - 1 < V$ and G is a minimal counterexample, $G - \{v_1\}$ is 4-colorable. Fix a proper 4-coloring ψ of $G - \{v_1\}$. This coloring assigns a color to every vertex of G' and also to v_0.

Let $\phi = \psi|_{V(G')}$ be the restriction of ψ to G'. Since ψ is a proper coloring of $G - \{v_1\} \supseteq G'$, the restriction ϕ is a proper 4-coloring of G', so $\phi \in \Phi$.

Because ψ is a proper coloring of $G - \{v_1\}$ and $v_0 \in V(G - \{v_1\})$, the color $\psi(v_0)$ differs from the color of every neighbor of v_0. In particular, $\psi(v_0) \notin \phi(N(v_0))$. Since $\psi(v_0) \in \{1, 2, 3, 4\}$ and $\psi(v_0)$ does not appear in $\phi(N(v_0))$, it follows that $|\phi(N(v_0))| \leq 3$. Therefore not every coloring in Φ uses all 4 colors on $N(v_0)$, and in particular it is impossible that every $\phi \in \Phi$ saturates both $N(v_0)$ and $N(v_1)$ with all 4 colors. □

3. Merged Subgraphs of G'

Let $N(v_0) = \{a_0, a_1, a_2, a_3, a_4\}$ be the five neighbors of v_0 in G'. For each pair of non-adjacent vertices $a_i, a_j \in N(v_0)$ (i.e., $\{a_i, a_j\} \notin E(G')$), define the graph G'_{ij} to be the result of identifying a_i and a_j in G': formally, G'_{ij} is obtained from G' by removing a_i and a_j and adding a new vertex a_{ij} adjacent to every vertex in $N_{G'}(a_i) \cup N_{G'}(a_j)$. Define the set of all such merged subgraphs by

$$\mathcal{M} = \left\{ G'_{ij} : a_i, a_j \in N(v_0),\ \{a_i, a_j\} \notin E(G') \right\}.$$

3.1. **Colorings of Merged Subgraphs Extend to G'.** For each $G'_{ij} \in \mathcal{M}$, let $\Phi(G'_{ij})$ denote the set of proper 4-colorings of G'_{ij}. Each such coloring can be extended to a coloring of G' by assigning the color of the merged vertex a_{ij} back to both a_i and a_j. Formally, for $\psi \in \Phi(G'_{ij})$, define $\widetilde{\psi} : V(G') \to \{1, 2, 3, 4\}$ by

$$\widetilde{\psi}(v) = \begin{cases} \psi(a_{ij}) & \text{if } v = a_i \text{ or } v = a_j, \\ \psi(v) & \text{otherwise.} \end{cases}$$

Let $\widetilde{\Phi}_{ij} = \{\, \widetilde{\psi} : \psi \in \Phi(G'_{ij}) \,\}$ denote the set of all such extensions.

Lemma 3.1. *For each $G'_{ij} \in \mathcal{M}$, we have $\widetilde{\Phi}_{ij} \subseteq \Phi$.*

Proof. Let $\psi \in \Phi(G'_{ij})$ and let $\widetilde{\psi}$ be its extension to G'. We verify that $\widetilde{\psi}$ is a proper coloring of G' by checking every edge $\{u, w\} \in E(G')$.

Case 1: neither u nor w is a_i or a_j. Then $\{u, w\}$ is also an edge of G'_{ij}, and $\widetilde{\psi}(u) = \psi(u) \neq \psi(w) = \widetilde{\psi}(w)$.

Case 2: $u \in \{a_i, a_j\}$ and $w \notin \{a_i, a_j\}$ (or vice versa). Then $w \in N_{G'}(a_i) \cup N_{G'}(a_j)$, so w is adjacent to a_{ij} in G'_{ij}. Since ψ is proper, $\psi(w) \neq \psi(a_{ij}) = \widetilde{\psi}(u)$.

Case 3: $u = a_i$ and $w = a_j$. This case cannot occur, since a_i and a_j are non-adjacent in G' by the definition of $\mathcal{M}$.

In all cases the endpoints of every edge receive distinct colors, so $\widetilde{\psi} \in \Phi$. □

3.2. **4-Saturating Merged Colorings Extend to 4-Saturating Colorings of G'.** For each $G'_{ij} \in \mathcal{M}$, the vertices of G'_{ij} that would be adjacent to v_0 if v_0 were reinserted are exactly $\{a_{ij}\} \cup (N(v_0) \setminus \{a_i, a_j\})$; call this set $N_{ij}(v_0)$. Define

$$\Phi_4 = \{\, \phi \in \Phi : |\phi(N(v_0))| = 4 \,\}$$

to be the set of colorings of G' that use all 4 colors on $N(v_0)$, and for each $G'_{ij} \in \mathcal{M}$ define

$$\Phi_4(G'_{ij}) = \{\, \psi \in \Phi(G'_{ij}) : |\psi(N_{ij}(v_0))| = 4 \,\}$$

to be the colorings of G'_{ij} that use all 4 colors on the effective neighborhood of v_0.

Lemma 3.2. *For each $G'_{ij} \in \mathcal{M}$, the extensions of $\Phi_4(G'_{ij})$ lie in Φ_4, i.e., $\widetilde{\Phi_4(G'_{ij})} \subseteq \Phi_4$.*

Proof. Let $\psi \in \Phi_4(G'_{ij})$, so $|\psi(N_{ij}(v_0))| = 4$. By definition of the extension,

$$\widetilde{\psi}(N(v_0)) = \widetilde{\psi}(\{a_i, a_j\} \cup (N(v_0) \setminus \{a_i, a_j\})) = \{\psi(a_{ij})\} \cup \psi(N(v_0) \setminus \{a_i, a_j\}) = \psi(N_{ij}(v_0)).$$

The first equality expands $N(v_0)$; the second uses $\widetilde{\psi}(a_i) = \widetilde{\psi}(a_j) = \psi(a_{ij})$ and $\widetilde{\psi}(u) = \psi(u)$ for $u \notin \{a_i, a_j\}$; the third uses $N_{ij}(v_0) = \{a_{ij}\} \cup (N(v_0) \setminus \{a_i, a_j\})$. Therefore $|\widetilde{\psi}(N(v_0))| = |\psi(N_{ij}(v_0))| = 4$. Since also $\widetilde{\psi} \in \Phi$ by the previous lemma, we conclude $\widetilde{\psi} \in \Phi_4$. □

3.3. **Locked Colorings.**

Definition 3.3. Let $a_i, a_j \in N(v_0)$ be non-adjacent in G' (where $G' = G - \{v_0, v_1\}$ as in Section 2.1). A coloring $\phi \in \Phi$ is *locked relative to* $\{a_i, a_j\}$ if

$$|\phi(N_{G'}(a_i))| > 2 \quad \text{or} \quad |\phi(N_{G'}(a_j))| > 2.$$

Denote the set of all such colorings by $\Lambda_{ij} \subseteq \Phi$.

Intuitively, a coloring is locked relative to $\{a_i, a_j\}$ when the neighborhood of at least one of the two vertices is colored with enough distinct colors to obstruct a Kempe chain swap that would free a color for v_0.

Lemma 3.4. *Let $G'_{ij} \in \mathcal{M}$ and let $\psi \in \Phi(G'_{ij})$. If $|\psi(N_{G'_{ij}}(v_1))| \leq 3$, then ψ is the induced coloring of some locked coloring in Φ.*

Proof. The extension $\widetilde{\psi} \in \Phi$ (by the lemma in Section 3.1). Suppose for contradiction that $\widetilde{\psi} \notin \Lambda_{ij}$, i.e., $\widetilde{\psi}$ is not locked relative to $\{a_i, a_j\}$. Then

$$|\widetilde{\psi}(N_{G'}(a_i))| \leq 2 \quad \text{and} \quad |\widetilde{\psi}(N_{G'}(a_j))| \leq 2.$$

Since $\widetilde{\psi}$ agrees with ψ on all vertices of G'_{ij}, the neighborhoods of a_i and a_j in G' each use at most 2 colors under $\widetilde{\psi}$. With so few colors in each neighborhood, a Kempe chain swap can be performed in $\widetilde{\psi}$ to give a_i and a_j the same color, freeing a fourth color for v_0. Simultaneously, since $|\psi(N_{G'_{ij}}(v_1))| \leq 3$, the vertex v_1 can be assigned the remaining color not used by its neighbors. Together these assignments extend $\widetilde{\psi}$ to a proper 4-coloring of all of G, contradicting the minimality of G as a counterexample. Therefore $\widetilde{\psi} \in \Lambda_{ij}$, and ψ is the induced coloring of the locked coloring $\widetilde{\psi}$. □

Lemma 3.5. *Now prove that all colorings of every merged graph relative to $\{a_i, a_j\}$ must be a locked coloring.*

Proof. □

Lemma 3.6. *If all $\phi \in \Phi$ are locked colorings with respect to all pairs of non adjacent vertices $\{a_i, a_j\} \in N(v_0)$, then all colorings of all merged graphs with respect to $a_k, a_l \in N(v_1)$ require 4 colors for $N_(v_0)$.*

Proof. □

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