AN ITERATED REDUCTION IN THE REDUCED DUAL

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Abstract.

1. Setup and background

This paper is a follow-up to *Face-Monochromatic Pairs and the Four Colour Theorem* [1], which introduced the reduced-dual construction: given a minimal counterexample G to the Four Colour Theorem, a degree-5 vertex v of G (equivalently a pentagonal face F_v of $G' = \mathrm{dual}(G)$), and an index $i \in \{0,1,2,3,4\}$, the *reduced dual* $\widehat{G}'_{v,i}$ is the cubic plane graph obtained from G' by deleting the five boundary vertices of F_v, listing the resulting five degree-2 vertices clockwise as $A_0, \dots, A_4$ along the new face F, attaching a new apex vertex v_n to A_i, A_{i+1}, A_{i+2} by three new edges, and adding the chord $A_{i+3}A_{i+4}$. The four edges added by steps (3) and (4) are named the *side*-0 *edge* (v_nA_i), the *spike edge* (v_nA_{i+1}), the *side*-1 *edge* (v_nA_{i+2}), and the *merged edge* ($A_{i+3}A_{i+4}$). The parent paper also proves two structural lemmas about every proper 3-edge-colouring φ of $\widehat{G}'_{v,i}$:

- the *chord-apex lemma*, asserting $\varphi(\text{spike}) = \varphi(\text{merged})$;
- the *Kempe-cycle lemma*, asserting that the spike and merged edge lie on a common $\{\varphi(\text{spike}), \varphi(\text{side}-j)\}$-Kempe cycle through the side-j edge for both $j = 0, 1$.

We refer the reader to [1] for the precise definitions, proofs, and the pentagonal-externals lemma we will reuse below.

2. The iterated reduction

The reduced-dual construction can be iterated: starting from a proper 3-edge-colouring φ_1 of a reduced dual $\widehat{G}'_{v,i}$, we apply the construction again to that graph at a pentagonal face whose ten incident edges avoid the four named edges from the first reduction, extending φ_1 across the new reduction. The protected edges accumulate into a set E that grows by four per iteration, and the process terminates when E has blocked every pentagonal face.

Algorithm 2.1 (Iterated reduction with protected edges)**.** Let G be a triangulation we assume to be a minimal counterexample to the Four Colour Theorem. The algorithm produces a sequence $H_1, H_2, \dots$ of cubic plane graphs, proper 3-edge-colourings φ_t of H_t, and a growing set E of protected edges.

(0) Form $G' := \mathrm{dual}(G)$, a cubic plane graph.

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(1) Choose a degree-5 vertex v of G (equivalently a pentagonal face F_v of G') and an index $i_1 \in \{0, \ldots, 4\}$. Apply the reduced-dual construction of [1] to form $H_1 := \widehat{G'}_{v,i_1}$, and fix any proper 3-edge-colouring φ_1 of H_1 (one exists by the minimality of G).
(2) Initialise $E := \{\text{spike}, \text{side-0}, \text{side-1}, \text{merged}\}$, the four named edges of the reduction in (1).
(3) (Iterate.) At step $t \geq 2$, given H_{t-1}, φ_{t-1}, and $E \subseteq E(H_{t-1})$:
    (a) Find a pentagonal face F of H_{t-1} whose ten incident edges — the five boundary edges of ∂F and the five external edges at ∂F — are all outside E. If no such F exists, terminate.
    (b) By the pentagonal-externals lemma of [1] applied to H_{t-1} at F under φ_{t-1}, the external vector has shape (a, b, c, c, c) up to cyclic rotation. Choose an index i_t for which $\varphi_{t-1}(f_{i_t+3}) = \varphi_{t-1}(f_{i_t+4})$ and $\varphi_{t-1}(f_{i_t})$, $\varphi_{t-1}(f_{i_t+1})$, $\varphi_{t-1}(f_{i_t+2})$ are three distinct colours.
    (c) Apply the reduced-dual construction of [1] to H_{t-1} at (F, i_t) to form H_t.
    (d) Extend φ_{t-1} to a proper 3-edge-colouring φ_t of H_t: every surviving edge keeps its φ_{t-1}-colour, and each new edge takes the unique colour completing the palette at its endpoint (consistent across both endpoints of the chord by the choice of i_t).
    (e) Add the four named edges of the step-t reduction to E.
(4) Repeat (3) until termination.

Remark 2.2. At each iteration, $|V(H_t)| = |V(H_{t-1})| - 4$ and $|E(H_t)| = |E(H_{t-1})| - 6$, so H_t shrinks at a fixed rate; the protected set $|E|$ grows by exactly four; and every protected edge survives all subsequent reductions. Since the graph is finite, termination is guaranteed. By the pentagonal-externals lemma of [1], step (b) never fails: some valid i_t always exists for any pentagonal face under any proper colouring. Termination is therefore combinatorial: it occurs precisely when E touches every pentagonal face of H_{t-1}.

Remark 2.3. The chord-apex lemma of [1] applies only at $t = 1$, when H_1 is a reduced dual of G'. For $t \geq 2$, H_t is a reduced dual of H_{t-1} rather than of G', and H_{t-1} is itself 3-edge-colourable, so the non-3-edge-colourability argument that drives the chord-apex lemma does not carry over. Whether the constraints accumulated in E propagate any further structure to φ_t for $t \geq 2$ is left open.

3. Structural lemmas on the algorithm's output

Lemma 3.1 (Exactly one matching pair in the algorithm's output)**.** *Let G be a minimal counterexample to the Four Colour Theorem, and let (H_{t^*}, φ_{t^*}) be the final graph-and-colouring produced by some terminating execution of Algorithm 2.1 on G, with named pairs* $(\text{spike}_t, \text{merged}_t)$ *for $t = 1, \ldots, t^*$. Then there is exactly one t with $\varphi_{t^*}(\text{spike}_t) = \varphi_{t^*}(\text{merged}_t)$, and it is $t = 1$.*

Proof. The algorithm never re-colours an existing edge: at each iteration step (3d) every surviving edge keeps its φ_{t-1}-colour, and the four new edges receive fresh colours forced by propriety. Hence for every $1 \leq k \leq t \leq t^*$,

$$\varphi_t(\text{spike}_k) = \varphi_k(\text{spike}_k), \qquad \varphi_t(\text{merged}_k) = \varphi_k(\text{merged}_k);$$

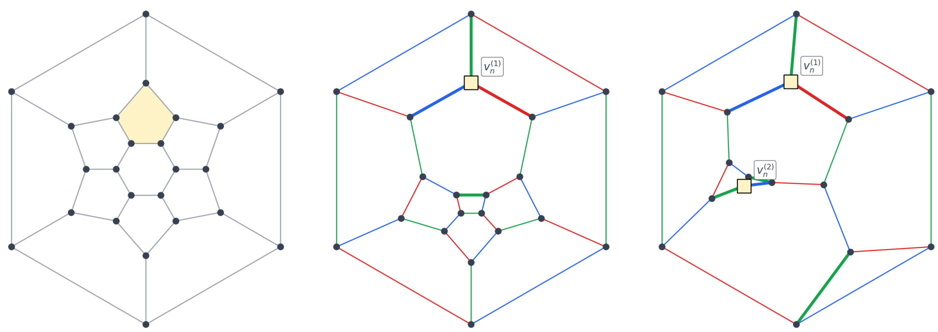

FIGURE 1. Algorithm 2.1 on $G' = \mathrm{dual}(G)$, where G is the first min-degree-5 plantri triangulation on 14 vertices and φ_1 is a specific proper 3-edge-colouring of H_1 that satisfies both the chord-apex and Kempe-cycle conditions of [1], found by `experiments/search_kempe_property.py`. *Left:* G' (24 vertices, 36 edges) with the chosen pentagonal face shaded. *Centre:* H_1 (20 vertices, 30 edges) after step (1) with $i_1 = 1$, 3-edge-coloured by φ_1; the four edges around $v_n^{(1)}$ in E are drawn thicker, and the spike and merged edges share the colour green. *Right:* H_2 (16 vertices, 24 edges) after step (3) with $i_t = 3$; eight edges are protected, and the algorithm terminates one step later (no remaining safe pentagonal face in H_2). The generating script is `experiments/draw_iterated_reduction_n14.py`; layouts are Tutte barycentric embeddings with the outer face picked to keep $v_n^{(1)}, v_n^{(2)}$ in the interior.

the colours of the step-k named edges, once written, are permanent. It suffices to compare $\varphi_k(\mathrm{spike}_k)$ and $\varphi_k(\mathrm{merged}_k)$ at the step where each pair is introduced.

Case $k = 1$**.** Since G is a minimal counterexample, H_1 is a reduced dual of G'. The chord-apex lemma of [1] applied to φ_1 gives $\varphi_1(\mathrm{spike}_1) = \varphi_1(\mathrm{merged}_1)$.

Case $k \geq 2$**.** At step k the algorithm picks an index i_k for which $f_{i_k+3} = f_{i_k+4}$ (chord consistency) and $f_{i_k}, f_{i_k+1}, f_{i_k+2}$ are pairwise distinct (propriety at the new v_n), where f is the external vector of the chosen pentagonal face of H_{k-1} under φ_{k-1}. Step (3d) then assigns

$$\varphi_k(\mathrm{spike}_k) = f_{i_k+1}, \qquad \varphi_k(\mathrm{merged}_k) = f_{i_k+3}.$$

By the pentagonal-externals lemma of [1], f has the $(2, 2, 1)$ pattern: a block of three consecutive positions $\{p, p+1, p+2\}$ (mod 5) on which it is constantly some colour c, while the remaining two positions $\{p+3, p+4\}$ hold the two non-c colours, one each. The condition $f_{i_k+3} = f_{i_k+4}$ forces (i_k+3, i_k+4) to be either $(p, p+1)$ or $(p+1, p+2)$ — the two consecutive pairs inside the block — and correspondingly $i_k + 1 \in \{p+3, p+4\}$, *outside* the block. So f_{i_k+1} is not c, whereas $f_{i_k+3} = c$, and hence $\varphi_k(\mathrm{spike}_k) \neq \varphi_k(\mathrm{merged}_k)$.

Combining the two cases, exactly one $t \in \{1, \ldots, t^*\}$ — namely $t = 1$ — has $\varphi_{t^*}(\mathrm{spike}_t) = \varphi_{t^*}(\mathrm{merged}_t)$. □

Lemma 3.2 (All-distinct colouring exists on a 4-colourable graph)**.** *Let G be a* 4*-colourable maximal planar graph of minimum degree* ≥ 5 *(equivalently, a maximal planar graph that is* not *a minimal counterexample to the Four Colour Theorem). Then there is an execution of Algorithm 2.1 on G whose final colouring φ_{t^*} satisfies $\varphi_{t^*}(\mathrm{spike}_t) \neq \varphi_{t^*}(\mathrm{merged}_t)$ for every $t \in \{1, \ldots, t^*\}$. In particular, there exists a proper* 3*-edge-colouring of H_{t^*} under which every spike-merged pair has distinct colours.*

Proof. The argument mirrors Lemma 3.1, but extends a colouring *downward* from G' rather than carrying one forward from H_1.

Since G is 4-colourable, by Tait's theorem $G' = \mathrm{dual}(G)$ admits a proper 3-edge-colouring ξ. Apply the pentagonal-externals lemma of [1] to ξ at the pentagonal face F_v selected in step (1): the external vector $f = (f_0, \ldots, f_4)$ at F_v under ξ has the $(3, 1, 1)$ cyclic-consecutive shape, with a block of three consecutive positions $\{p, p+1, p+2\}$ (mod 5) holding a common colour c, and the remaining two positions $\{p + 3, p + 4\}$ holding the two non-c colours, one each. The algorithm's choice of i_1 forces $\{i_1 + 3, i_1 + 4\}$ inside the c-block (so the chord is consistently coloured) and the three positions $\{i_1, i_1 + 1, i_1 + 2\}$ pairwise distinct; in particular $i_1 + 1$ lies *outside* the c-block.

Choose φ_1 to be the proper 3-edge-colouring of H_1 that agrees with ξ on every surviving edge and assigns each new edge at A_j the unique third colour at A_j. Then $\varphi_1(\mathrm{spike}_1) = f_{i_1+1}$, a value not equal to c, while $\varphi_1(\mathrm{merged}_1) = f_{i_1+3} = c$, so $\varphi_1(\mathrm{spike}_1) \neq \varphi_1(\mathrm{merged}_1)$.

The same argument repeats at every step $k \geq 2$: the external vector at the chosen pentagonal face under φ_{k-1} has the $(3, 1, 1)$ cyclic-consecutive shape (pentagonal-externals lemma of [1]), the algorithm's index choice i_k puts $i_k + 3, i_k + 4$ inside the colour block and $i_k + 1$ outside, and step (3d) thus assigns $\varphi_k(\mathrm{spike}_k) \neq \varphi_k(\mathrm{merged}_k)$. The algorithm preserves these colours through every later step, so $\varphi_{t^*}(\mathrm{spike}_t) \neq \varphi_{t^*}(\mathrm{merged}_t)$ for every $t \in \{1, \ldots, t^*\}$. $\square$

References

[1] E. Bauerfeld, *Face-Monochromatic Pairs and the Four Colour Theorem.* Companion paper.