COLORED PENTAGON REDUCTIONS

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Abstract. We investigate specific types of minors we can obtain by a sequence of contractions of two incident to a sequence of vertices of degree 5 relative to a proper coloring.

1. Definitions

Definition 1.1. Let G be a maximal planar graph with a proper 4-coloring $c : V(G) \to \{1,2,3,4\}$. A *colored pentagon reduction* is a function

$$R : (G, c, v_0, v_1, v_2) \mapsto G/\{v_1 v_0,\ v_2 v_0\}$$

where $v_0 \in V(G)$ has degree 5, v_1 and v_2 are adjacent vertices satisfying $c(v_1) = c(v_2)$, and $G/\{v_1 v_0,\ v_2 v_0\}$ denotes the minor obtained by contracting v_1 and v_2 each into v_0.

Lemma 1.2. *If $G' = R(G, c, v_0, v_1, v_2)$, then G' has a proper 4-coloring.*

Proof. The idea is to recolor v_0 with the shared color of v_1 and v_2, and check that this creates no new conflicts.

Let $\alpha = c(v_1) = c(v_2)$ and define $c' : V(G') \to \{1,2,3,4\}$ by

$$c'(v) = \begin{cases} \alpha & \text{if } v = v_0, \\ c(v) & \text{otherwise.} \end{cases}$$

Every edge of G' not incident to v_0 is an edge of G between vertices outside $\{v_1, v_2\}$, so those edges are still properly colored. It remains to check edges $v_0 w$ in G', i.e., that no neighbor of v_0 in G' also has color α.

Since $c(v_1) = c(v_2)$ and c is proper, v_1 and v_2 cannot be adjacent in G, so they are non-adjacent in the 5-cycle induced by $N_G(v_0)$ (which exists because G is maximal planar). One can check that in any 5-cycle, two non-adjacent vertices together cover every other vertex — that is, every remaining vertex is adjacent to at least one of them. Therefore every $w \in N_G(v_0) \setminus \{v_1, v_2\}$ is adjacent in G to v_1 or v_2, so $c(w) \neq \alpha = c'(v_0)$.

For new neighbors of v_0 coming from the contraction, namely $w \in (N_G(v_1) \cup N_G(v_2)) \setminus \{v_0, v_1, v_2\}$, we get $c(w) \neq \alpha = c'(v_0)$ for free since c is proper on G.

Hence c' is a proper 4-coloring of G'. □

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