COLORED EDGE FLIP CLASSES

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Abstract. Suppose the Four Color Theorem fails, and let G_0 be a maximal planar graph of minimum order with $\chi(G_0) \geq 5$. Using an edge-deletion argument together with a Kempe-chain swap, we show that every graph in the flip neighborhood $\mathcal{N}(G_0)$ — the set of maximal planar graphs obtainable from G_0 by a single admissible edge flip — is 4-colorable. We also introduce the colored edge flip class $\mathcal{C}(H, \varphi)$ of a maximal planar graph H and a proper 4-coloring φ of H, and record that $G_0 \notin \mathcal{C}(H, \varphi)$ for any $H \in \mathcal{N}(G_0)$ and any such φ.

1. Motivation

The Four Color Theorem asserts that every planar graph is properly 4-colorable, or equivalently that no maximal planar graph G satisfies $\chi(G) \geq 5$. Suppose, towards a contradiction, that such a graph exists; let G_0 be one of minimum order. Any structural property shared by every maximal planar graph H with $|V(H)| = |V(G_0)|$ is then automatically inherited by G_0, and any property *not* satisfied by G_0 excludes a portion of the class of maximal planar graphs from playing the role of a minimum counterexample.

Our principal observation (Theorem 4.4) is that every graph in the *flip neighborhood* of G_0 — the set $\mathcal{N}(G_0)$ of maximal planar graphs obtainable from G_0 by a single admissible edge flip — is 4-colorable. In other words, G_0 sits at the boundary of 4-colorability: a single flip in any direction yields a 4-colorable graph.

To track this rigidity at the level of individual 4-colorings, we introduce the *colored edge flip class* $\mathcal{C}(H, \varphi)$ of a maximal planar graph H and a proper 4-coloring φ of H: the set of maximal planar graphs reachable from H by sequences of admissible edge flips that each preserve φ. Theorem 4.5 records that $G_0 \notin \mathcal{C}(H, \varphi)$ for any $H \in \mathcal{N}(G_0)$ and any proper 4-coloring φ of H.

2. Preliminaries

Let G be a maximal planar graph with $|V(G)| \geq 4$, embedded in the plane so that every face — including the outer face — is a triangle. Every edge $uv \in E(G)$ is then shared by exactly two triangular faces uvw and uvx whose union is a quadrilateral $uwvx$ with diagonal uv.

Definition 2.1 (Edge flip)**.** Let G be a maximal planar graph and let $uv \in E(G)$ be an edge whose two incident triangular faces are uvw and uvx. The *edge flip* (or *diagonal flip*) at uv is the operation that deletes the edge uv and inserts the edge wx in its place, replacing the two triangles uvw and uvx by the two triangles uwx

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and vwx. The flip is *admissible* if $wx \notin E(G)$; otherwise the resulting multigraph is not simple and the flip is forbidden.

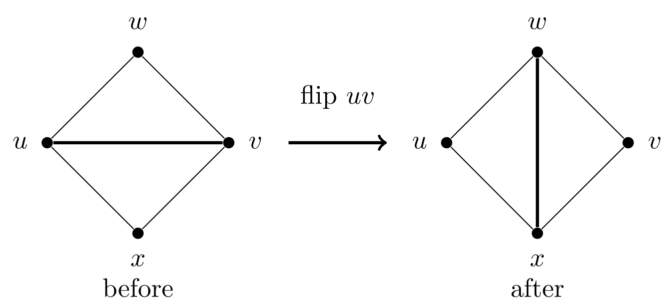

FIGURE 1. An edge flip replaces the diagonal uv of the quadrilateral $uwvx$ with the diagonal wx.

3. Flip neighborhoods and colored edge flip classes

For a maximal planar graph G and an admissible edge $uv \in E(G)$ with incident triangles uvw, uvx, write

$$G^{\mathrm{flip}(uv)} \;=\; \big(V(G),\, (E(G) \setminus \{uv\}) \cup \{wx\}\big)$$

for the graph obtained from G by flipping uv.

Definition 3.1 (Flip neighborhood)**.** Let G be a maximal planar graph. The *flip neighborhood* of G is the set

$$\mathcal{N}(G) \;=\; \big\{\, G^{\mathrm{flip}(uv)} : uv \in E(G) \text{ and the flip at } uv \text{ is admissible} \,\big\}$$

of maximal planar graphs obtainable from G by a single admissible edge flip.

Definition 3.2 (Colored edge flip class)**.** Let G be a maximal planar graph and let φ be a proper 4-coloring of G. The *colored edge flip class* of (G, φ) is the set $\mathcal{C}(G, \varphi)$ of maximal planar graphs reachable from G by some (possibly empty) sequence of admissible edge flips, each of which leaves φ a proper 4-coloring of the resulting graph. Explicitly, $H \in \mathcal{C}(G, \varphi)$ iff there exist graphs $G = G_0, G_1, \ldots, G_k = H$ such that for each $0 \le i < k$, $G_{i+1} = G_i^{\mathrm{flip}(u_i v_i)}$ for some $u_i v_i \in E(G_i)$ whose flip is admissible in G_i and whose opposite vertices w_i, x_i satisfy $\varphi(w_i) \neq \varphi(x_i)$.

4. The flip neighborhood of a minimum-order counterexample

Definition 4.1 (Edge-deletion subgraph)**.** Let G be a maximal planar graph and $uv \in E(G)$. The *edge-deletion subgraph at uv* is the spanning subgraph $G - uv = (V(G),\, E(G) \setminus \{uv\})$. Write $\mathcal{D}(G) = \{G - uv : uv \in E(G)\}$.

Lemma 4.2. *Let G_0 be a maximal planar graph of minimum order with $\chi(G_0) \ge 5$. Then every $H \in \mathcal{D}(G_0)$ is 4-colorable.*

Proof. Fix $uv \in E(G_0)$ and let G_0/uv denote the simple planar graph obtained by contracting uv and discarding parallel edges. Then G_0/uv is a simple planar graph on $|V(G_0)| - 1 \geq 4$ vertices but is not in general a triangulation; triangulate its planar embedding (by adding chords inside any non-triangular face) to obtain a maximal planar graph T on the same vertex set, with G_0/uv as a spanning subgraph and $|V(T)| < |V(G_0)|$. By the minimality of G_0, T admits a proper 4-coloring, which restricts to a proper 4-coloring c of G_0/uv. Let z be the contracted vertex and define $c' \colon V(G_0) \to \{1, 2, 3, 4\}$ by $c'(u) = c'(v) = c(z)$ and $c'(y) = c(y)$ for $y \notin \{u, v\}$. Every edge of $G_0 - uv$ is either disjoint from $\{u, v\}$ or incident to exactly one of them; in either case the corresponding edge of G_0/uv has distinct endpoints under c, so c' assigns its endpoints distinct colors. The edge uv itself is absent from $G_0 - uv$, so c' is a proper 4-coloring of $G_0 - uv$. □

Lemma 4.3. *Let G_0 be a maximal planar graph of minimum order with $\chi(G_0) \geq 5$, fix $uv \in E(G_0)$, and let φ be any proper 4-coloring of $G_0 - uv$. Write $a = \varphi(u)$ and let b, c, d denote the three remaining colors. Then:*

(1) *$\varphi(v) = a$;*
(2) *the subgraph of $G_0 - uv$ induced by the vertices of color a or b contains a path from u to v;*
(3) *the subgraph of $G_0 - uv$ induced by the vertices of color a or c contains a path from u to v.*

Proof. (1) If $\varphi(v) \neq a$ then φ is already a proper 4-coloring of G_0, since the only edge of G_0 absent from $G_0 - uv$ is uv and its endpoints have distinct colors. This contradicts $\chi(G_0) \geq 5$, so $\varphi(v) = a$.

(2) Suppose, for contradiction, that u and v lie in distinct connected components of the subgraph of $G_0 - uv$ induced by the color classes a and b. Let C be the component containing u, and define $\varphi' \colon V(G_0) \to \{a, b, c, d\}$ by swapping colors $a \leftrightarrow b$ on C and leaving every other vertex unchanged. Then φ' is a proper 4-coloring of $G_0 - uv$ with $\varphi'(u) = b$ and $\varphi'(v) = a$, contradicting part (1) applied to φ'.

(3) Identical to (2) with c in place of b. □

Theorem 4.4. *Let G be a minimum-order maximal planar graph with $\chi(G) \geq 5$. Then every $H \in \mathcal{N}(G)$ is 4-colorable.*

Proof. Fix an edge $e = uv \in E(G)$, and let F_0, F_1 be the two triangular faces of G incident to e, so that $\{w, x\} = \big(V(F_0) \cup V(F_1)\big) \setminus \{u, v\}$. By Lemma 4.2, $G - e$ admits a proper 4-coloring φ.

Case 1: $\varphi(w) \neq \varphi(x)$. Then φ is also a proper 4-coloring of the graph induced by the edge flip of e.

Case 2: $\varphi(w) = \varphi(x)$. Set $a = \varphi(u)$; by Lemma 4.3(1), $\varphi(v) = a$ as well, and the edges $uw, vw \in E(G - e)$ force $\varphi(w) \neq a$. Choose a color $b \notin \{a, \varphi(w)\}$. By Lemma 4.3, there is a path P from u to v in the subgraph of $G - e$ induced by the vertices of color a or b. Let $\{c, d\} = \{1, 2, 3, 4\} \setminus \{a, b\}$; then $\varphi(w) = \varphi(x) \in \{c, d\}$.

Any path from w to x in the subgraph of $G - e$ induced by the vertices of color c or d would, in the plane embedding of $G - e$, cross P; but its vertices have colors in $\{c, d\}$ and the vertices of P have colors in $\{a, b\}$, and these sets are disjoint, so the two paths share no vertex. Hence w and x lie in distinct connected components of the $\{c, d\}$-colored subgraph of $G - e$. Swapping colors $c \leftrightarrow d$ on the component containing w yields a proper 4-coloring of $G - e$ in which $\varphi(w) \neq \varphi(x)$, reducing to Case 1. $\square$

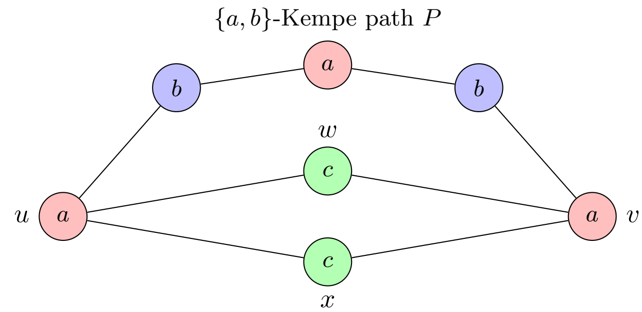

FIGURE 2. Case 2 of the proof of Theorem 4.4: u, v share color a and w, x share color c. The $\{a, b\}$-Kempe path P from u to v separates w from x in the plane, so no $\{c, d\}$-path between w and x can avoid crossing P; since the color sets $\{a, b\}$ and $\{c, d\}$ are disjoint, no such path exists.

Theorem 4.5. *Let G be a minimum-order maximal planar graph with $\chi(G) \geq 5$. Then for every $H \in \mathcal{N}(G)$ and every proper 4-coloring φ of H,*

$$G \notin \mathcal{C}(H, \varphi).$$

Proof. Suppose, for contradiction, that $G \in \mathcal{C}(H, \varphi)$ for some $H \in \mathcal{N}(G)$ and some proper 4-coloring φ of H. By Definition 3.2, there exists a sequence of maximal planar graphs $H = H_0, H_1, \ldots, H_k = G$ in which each H_{i+1} is obtained from H_i by an admissible edge flip that leaves φ a proper 4-coloring of H_{i+1}. By induction on i, φ is a proper 4-coloring of every H_i; in particular, φ is a proper 4-coloring of $H_k = G$. But $\chi(G) \geq 5$ admits no such coloring, a contradiction. $\square$