

LEVEL SWITCHING

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ABSTRACT.

1. INTRODUCTION

2. DEFINITIONS

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G . We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source). A *level source* of G is either:

- a face F of G (all vertices of F are level-0 sources), or
- a vertex v of degree 3 (the singleton $\{v\}$ is a level-0 source).

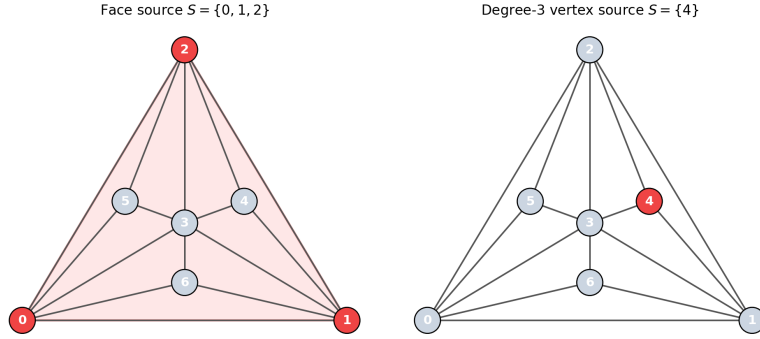


FIGURE 1. The two kinds of level source on a 7-vertex triangulation T (K_4 with vertices 4, 5, 6 stacked into the three interior faces). Left: the face source $S = \{0, 1, 2\}$ (level-0 vertices are the corners of the highlighted triangle). Right: the degree-3 vertex source $S = \{4\}$.

Definition 2.2 (Levels). Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \text{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 2.3 (Level cycle). A *level cycle* of G (with respect to a level source S) is a simple cycle in G all of whose vertices have the same level.

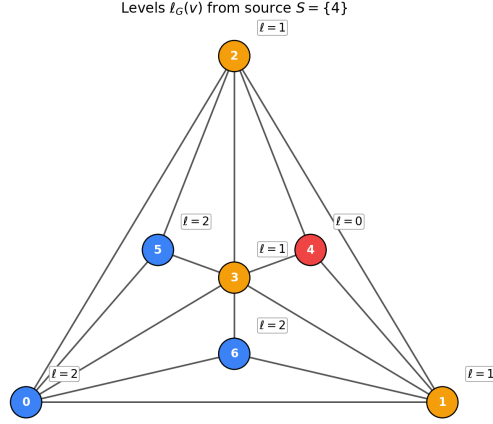


FIGURE 2. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

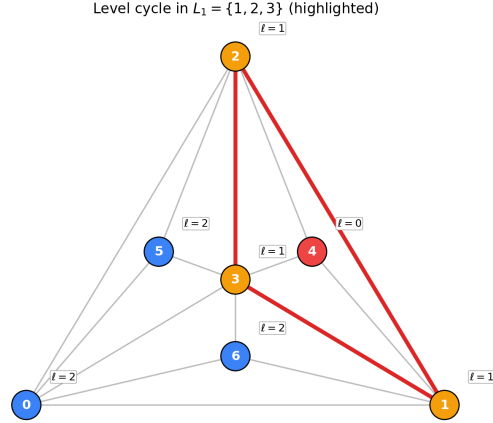


FIGURE 3. A level cycle in the triangulation of Figure 2. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

Definition 2.4 (Edge switch). Let G be a triangulation with level source S , and let $e = uv$ be an edge of a level cycle of G . The *edge switch* at e is the edge flip on e : writing uvw and uvx for the two triangular faces of G containing e , the edge uv is removed and the edge wx is added. As with any edge flip, the result is a triangulation on the same vertex set provided w and x are non-adjacent in G .

Definition 2.5 (Parity subgraph). Let G be a triangulation with level source S , and let G' be a triangulation on the same vertex set as G . The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod{2}\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G .

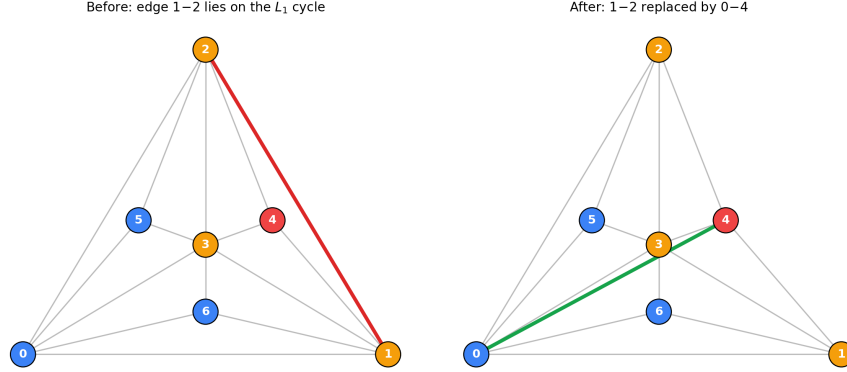


FIGURE 4. An edge switch on the level cycle of Figure 3. The chosen cycle edge $1-2$ is shared by the triangular faces $(0, 1, 2)$ and $(1, 2, 4)$; the switch deletes $1-2$ (red, left) and inserts $0-4$ (green, right). Vertex colours indicate the original levels in G .

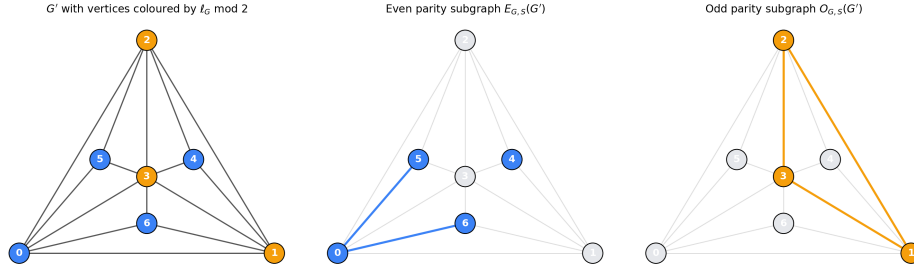


FIGURE 5. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 2 (here we take $G = G' = T$). Left: T with vertices coloured by $\ell_G \bmod 2$ (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0, 4, 5, 6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1, 2, 3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of G' .

Definition 2.6 (Facial depth). Let L_k be drawn with the outerplanar embedding inherited from Π_G , let D be the dual graph of this drawing with the outer face removed, and let \mathcal{B} be the set of inner faces of L_k whose bounding level cycle contains at least one edge of the outer cycle of L_k . The *facial depth* of an inner face F of L_k is

$$\text{depth}(F) = \min_{F' \in \mathcal{B}} \text{dist}_D(F, F'),$$

with the convention $\text{depth}(F) = \infty$ if no such F' exists. An inner face is *isolated* if $\text{depth}(F) \geq 1$.

Definition 2.7 (Surface switch). A *surface switch* is an edge switch (Definition 2.4) applied to an edge incident to two level cycles, one of facial depth d and the other of facial depth $d - 1$.

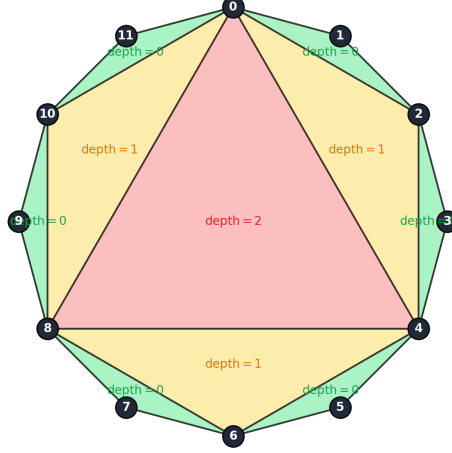
Facial depth in an outerplanar L_k 

FIGURE 6. Facial depths in a maximal outerplanar graph on 12 vertices. The six green ear-triangles share an edge with the outer 12-cycle and so lie in \mathcal{B} (depth 0). The three yellow “in-between” triangles $(0, 2, 4)$, $(4, 6, 8)$, $(8, 10, 11)$ have only diagonal edges but each is dual-adjacent to ears, giving them depth = 1. The central triangle $(0, 4, 8)$ is also all-diagonal; its dual neighbours are the three depth-1 triangles, so it is isolated with depth = 2.

Definition 2.8 (Balanced surface switch). Let σ be a surface switch on an edge $e = uv$ separating an inner face F of L_k of depth $d \geq 1$ from an adjacent inner face $F' = uvx$ of depth $d - 1$. We say σ is *balanced* if each of the two edges of $\partial F'$ other than uv (namely ux and vx) either lies on the outer cycle of L_k or is shared with an inner face of L_k of depth $d - 2$.

When $d = 1$ the condition reduces to “both ux and vx lie on the outer cycle of L_k ”, because no inner face has depth -1 ; in that case F' is a triangular “ear” hanging off uv .

3. OUTERPLANARITY OF LEVEL COMPONENTS

For each integer $k \geq 0$ and each (G, S) , write L_k for the subgraph of G induced by the level- k vertices. A *level component* of G (with respect to S) is a connected component of some L_k .

Theorem 3.1. *For every plane triangulation G and every level source S of G , every level component of G is outerplanar.*

Proof. Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph L_k is outerplanar. For $k = 0$, L_0 is either a single vertex (when S is a degree-3 vertex) or the triangle bounding the source face (when S is a face), both outerplanar.

Fix $k \geq 1$ and let D_k be the drawing of L_k inherited from Π_G . Let F^* be the face of D_k containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on ∂F^* , so u lies on the boundary of some other face of D_k . Take any path P in G from $v_0 \in S$ to u . As a curve in Π_G , P starts in F^* and ends at a point off ∂F^* , so it must transition from F^* to a different face of D_k ; in a planar embedding this can happen only at a vertex of D_k , that is, at a level- k vertex w on P . Either $w \neq u$ (so P has length $\geq \text{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every S -to- u path has length $\geq k + 1$, $\text{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. \square

Lemma 3.2. *Let C be a level component of G with respect to S , drawn with the outerplanar embedding inherited from Π_G , and let D be its inner-face dual. If F is an inner face of C with $\text{depth}(F) = d > 0$, then F is dual-adjacent to an inner face F' with $\text{depth}(F') = d - 1$.*

Proof. By Theorem 3.1, C is outerplanar, so the inner-face dual D is a forest (a standard fact; a tree when C is 2-connected).

Each leaf F_ℓ of D contains a single interior edge of C , so the remaining edges of ∂F_ℓ lie on the outer cycle of C . In particular F_ℓ has at least one outer-cycle edge, so $F_\ell \in \mathcal{B}$ and $\text{depth}(F_\ell) = 0$. Hence every tree component of D contains an element of \mathcal{B} , so the depths of all of its vertices are finite.

Choose a shortest path $F = F_0, F_1, \dots, F_d = F^*$ in D from F to some $F^* \in \mathcal{B}$ realising $\text{depth}(F) = d$. The suffix F_1, \dots, F_d witnesses $\text{depth}(F_1) \leq d - 1$. If $\text{depth}(F_1) \leq d - 2$, prepending the edge $F F_1$ to a witnessing path would give $\text{depth}(F) \leq d - 1$, contradicting $\text{depth}(F) = d$. Hence $\text{depth}(F_1) = d - 1$, and we may take $F' := F_1$. \square

Proposition 3.3. *Let σ be a balanced surface switch on the edge $e = uv$ separating F (depth $d \geq 1$) from $F' = uvx$ (depth $d - 1$), and let $G' = G - uv + wx$ be the result of the underlying edge flip, with uvw, uvx the two triangular faces of G at uv . Then in L'_k (the level- k subgraph of G' , with the level assignment of G):*

- (1) *the level cycle ∂F is destroyed; and*
- (2) *one or two new inner faces appear in L'_k , each of depth exactly $d - 1$.*

Proof. The flip removes uv from G , so ∂F is no longer a cycle of L'_k , proving (1). For (2) we split on whether the new edge wx re-enters L_k .

Case (i): $\{w, x\} \not\subseteq L_k$. Then $L'_k = L_k - uv$. The faces F and F' merge into a single new inner face \tilde{F} with boundary $(\partial F \cup \partial F') \setminus \{uv\}$. The dual neighbours of \tilde{F} in L'_k are exactly the former neighbours of F and F' other than each other; in particular they include all inner faces previously adjacent to F' across ux or vx , whose depths are at most $d - 2$ by Lemma 3.2 applied to F' (when $d \geq 2$). Thus $\text{depth}(\tilde{F}) \leq d - 1$.

For the matching lower bound, every neighbour of \tilde{F} has depth $\geq d - 2$ (neighbours inherited from F have depth $\geq d - 1$; neighbours inherited from F' have depth $\geq d - 2$). When $d \geq 2$, neither F nor F' has an outer-cycle edge, so neither does \tilde{F} , giving $\text{depth}(\tilde{F}) \geq d - 1$. When $d = 1$, $F' \in \mathcal{B}$ and its outer-cycle edge (necessarily distinct from the interior edge uv) survives on $\partial \tilde{F}$, so $\tilde{F} \in \mathcal{B}'$ and $\text{depth}(\tilde{F}) = 0 = d - 1$. In either case $\text{depth}(\tilde{F}) = d - 1$, giving the unique new face required by (2).

Case (ii): $\{w, x\} \subseteq L_k$. Then $F = uvw$ and $F' = uvx$ are triangular faces of L_k , and $L'_k = L_k - uv + wx$. The chord wx splits the quadrilateral $\partial(F \cup F')$ into two triangular faces $A = uwx$ and $B = vwx$ of L'_k . We show $\text{depth}(A) = \text{depth}(B) = d - 1$.

By symmetry it suffices to handle A . The dual neighbours of A in L'_k are A_{uw} (the inner face across uw , unchanged from L_k), A_{ux} (the inner face across ux , unchanged), and B (across the new edge wx). By balancedness of σ applied to the edge ux :

- if ux lies on the outer cycle of L_k , it remains on the outer cycle of L'_k , so $A \in \mathcal{B}'$ and $\text{depth}(A) = 0$ (which equals $d - 1$ because the balanced-with-outer-cycle case forces $d = 1$); or
- if A_{ux} is an inner face, balancedness gives $\text{depth}(A_{ux}) = d - 2$, so $\text{depth}(A) \leq 1 + (d - 2) = d - 1$.

For the lower bound in the second sub-case ($d \geq 2$): A 's edges are uw (an edge of F , interior because F has depth $d \geq 1$), wx (new, not on the outer cycle), and ux (interior in this sub-case), so $A \notin \mathcal{B}'$. Moreover every neighbour of A has depth $\geq d - 2$: A_{uw} inherits depth $\geq d - 1$ from being a former neighbour of F , A_{ux} has depth $d - 2$, and B has depth $\geq d - 2$ by the same argument applied symmetrically. Therefore $\text{depth}(A) \geq d - 1$, and combined with the upper bound, $\text{depth}(A) = d - 1$. \square

When does a balanced surface switch exist? For a chord uv of a maximal outerplanar graph, the *span* of uv is the minimum, over the two arcs from u to v on the outer cycle, of the number of outer-cycle vertices strictly between them.

Observation 3.4. For $d = 1$, an inner face F admits a balanced surface switch on some edge iff at least one edge of F has span 1 in the outer cycle of L_k . The opposite triangle across that edge – using the single outer-cycle vertex between its endpoints – is then an ear of F in \mathcal{B} , satisfying the $d = 1$ form of Definition 2.8.

The smallest maximal-outerplanar configuration violating this is a 9-vertex outer cycle triangulated so that the unique interior face $F = (0, 3, 6)$ has spans $(2, 2, 2)$ on its three edges (Figure 7). Each depth-0 neighbour of F carries exactly one outer-cycle edge, not two, so none qualifies as an ear of F ; no balanced surface switch is available.

Preprocessing toward balanced switches. When F has depth $d \geq 1$ but admits no balanced surface switch, perform a single (unbalanced) surface switch on any edge of F shared with a depth- $(d - 1)$ neighbour. By Proposition 3.3 the result is at least one new depth- $(d - 1)$ face; in Case (ii) it is accompanied by a new depth- d face A that replaces F as the next candidate. The hope is that the resulting A admits a balanced surface switch, or that iterating the preprocessing eventually exposes one.

Example 3.5. On the 9-vertex example, the (unbalanced) surface switch on edge $uv = 03$ – with $F' = (0, 2, 3)$, third vertex $x = 2$, and $w = 6$ – flips $03 \mapsto 26$ in G and produces $A = (0, 2, 6)$ at depth 1. The new face has spans $(1, 3, 2)$ on its edges, and the ear $(0, 1, 2)$ across the span-1 edge 02 is now a balanced surface-switch target on A (Figure 8).

We do not have a general termination theorem. The natural candidate monovariant for $d = 1$ is the minimum span among edges of the current depth-1 face that

Depth-1 face with no balanced surface switch

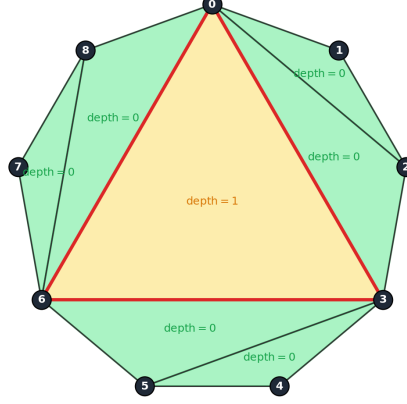


FIGURE 7. 9-vertex maximal outerplanar L_k . $F = (0, 3, 6)$ has depth = 1 and all three of its edges have span 2, so none of F 's depth-0 neighbours is an ear. No balanced surface switch is available on F .

Before: $F=(0,3,6)$ depth 1; spans (2,2,2) so no ear neighbour

After non-balanced switch $03 \rightarrow 26$: $A=(0,2,6)$ depth 1; edge 02 has span 1

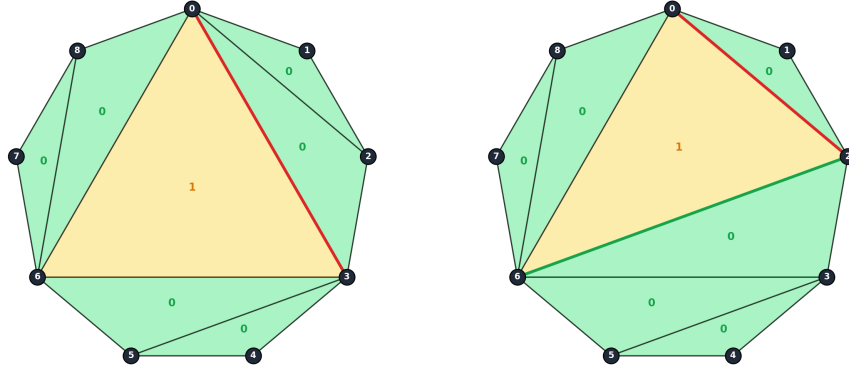


FIGURE 8. One step of preprocessing on the 9-vertex example. Left: $F = (0, 3, 6)$ has no edge of span 1; the chosen surface-switch edge $uv = 03$ (red) is unbalanced. Right: after the switch $03 \mapsto 26$ (green), the new depth-1 face $A = (0, 2, 6)$ has its edge 02 (red) at span 1, exposing the ear $(0, 1, 2)$ as a balanced surface-switch target.

are shared with a depth-0 neighbour; in Example 3.5 this drops from 2 to 1 in a single step. Whether such a monovariant strictly decreases under every unbalanced

surface switch – and a corresponding statement for $d \geq 2$, where balancedness depends on depth- $(d - 2)$ structure rather than just spans – remains open.

Question 3.6. Does iterated preprocessing reach a balanced surface switch in finitely many steps from every initial configuration? Equivalently, is there a monovariant on the inner-face structure of L_k that strictly decreases at every unbalanced surface switch on a maximum-depth face?