Empirical findings on the König-lift conjecture (Conj. *t2-induces-partition* from `worst_case_proof_sketch.tex`)

What was tested

The König-lift approach to chain pigeonhole (`worst_case_proof_sketch.tex`) conjectures that for an SP tire T_2 with $B_{\text{in}}^{(2)}$-chord structure such that every $O^{(2)}$-face has exactly 3 $B_{\text{in}}^{(2)}$-edges, there is an *induced* face partition $\widetilde{\mathcal{F}_2}$ of γ into triples with

$$\pi_U(\mathcal{C}(T_2)) \supseteq \mathcal{L}(\gamma, \widetilde{\mathcal{F}_2}).$$

The candidate construction (worst-case note, "one concrete attempt") groups γ-edges by the $O^{(2)}$-face whose D-triangle is cyclically next (or previous) on T'_{ann}.

Script: `experiments/induced_partition.py`.

Findings

$k = \lvert\gamma\rvert$	$k_2 = \lvert B_{\text{in}}^{(2)}\rvert$	chords on $B_{\text{in}}^{(2)}$	$\lvert\pi_U\rvert$	3^k	candidate 1 OK?	candidate 2 OK?
6	6	$(0,3)$ (faces 3+3)	90	729	**Yes**	**Yes**
6	6	$(0,2)(3,5)$ (faces 2+2+2)	456	729	N/A (not triples)	N/A
6	3	none (face 3, single)	84	729	N/A (not triples)	N/A
9	9	$(0,3)(3,6)$ (faces 3+3+3)	978	19683	**No**	**No**

(a) The candidate works at $k = 6$

For $k = k_2 = 6$ antipodal chord (the symmetric all-3-faces case), both candidate 1 (next-D) and candidate 2 (prev-D) produce the same kind of γ-partition (a two-block partition of size 3 each), and the Latin subset $\mathcal{L}(\gamma, \widetilde{\mathcal{F}_2})$ of size 36 is verified $\subseteq \pi_U$.

In fact *every* two-block triple-partition of $\{0, \dots, 5\}$ (there are 10 such) has Latin subset $\subseteq \pi_U$ here, because $\lvert\pi_U\rvert = 90$ is much larger than 36 and absorbs all of them.

(b) The candidate ***fails*** at $k = 9$

This is the surprise. For $k = k_2 = 9$ with chords $(0, 3), (3, 6)$ producing three $O^{(2)}$-faces of size 3 each:

- Candidate 1 (next-D): $\widetilde{\mathcal{F}_2} = \{\{0, 1, 8\},\ \{2, 3, 4\},\ \{5, 6, 7\}\}$. $\lvert\mathcal{L}\rvert = 216$, but $\mathcal{L} \nsubseteq \pi_U$ (some Latin elements are missing).
- Candidate 2 (prev-D): $\widetilde{\mathcal{F}_2} = \{\{0, 1, 2\},\ \{3, 4, 5\},\ \{6, 7, 8\}\}$. $\lvert\mathcal{L}\rvert = 216$, but $\mathcal{L} \nsubseteq \pi_U$.

- Of all 280 possible triple-partitions of $\{0, \ldots, 8\}$, only 8 have $\mathcal{L} \subseteq \pi_U$.

The eight surviving partitions are not contiguous blocks. Examples: $\{\{0, 2, 3\}, \{1, 6, 8\}, \{4, 5, 7\}\}$, $\{\{0, 2, 4\}, \{1, 6, 8\}, \{3, 5, 7\}\}$, etc. They do not have an obvious geometric interpretation in terms of T_2's annular triangulation.

(c) The asymmetric case ($k \neq k_2$) is outside scope

For $k = 6, k_2 = 3$ (the configuration with $T_2 = (3, -, \mathrm{SP})$), the candidate construction collapses to a single block of 6 γ-edges (since there is only one $O^{(2)}$-face), so it is not a triple-partition. Moreover, *no* triple-partition of $\{0, \ldots, 5\}$ has Latin subset $\subseteq \pi_U$ here.

So Conj. *t2-induces-partition* as currently stated does not cover $k \neq k_2$, and the empirical data shows there is no "rescue" partition of any kind.

Implications

The König lift's natural construction breaks past $k = 6$

The candidate $\widetilde{\mathcal{F}_2}$ from the worst-case note is the geometrically natural one (group γ-edges by their nearest $O^{(2)}$-face D-triangle), and it succeeds at $k = 6$ partly by coincidence: $|\pi_U|$ is so large that every triple-partition fits. At $k = 9$ the gap between $|\pi_U|$ and 3^k widens, and the candidate's specific partition is no longer in the small set of "correct" partitions.

The fact that only 8/280 partitions work at $k = 9$ suggests that whatever the right $\widetilde{\mathcal{F}_2}$ is, it is *not* just a function of T_2's outerplanar face structure — it must encode finer information about the annular triangulation.

Asymmetric pairs not covered at all

The empirical worst-case overlap $|S_1 \cap S_2| = 6$ in step-2 data comes from *asymmetric* pairs (e.g. $T_1 = (6, (0, 3), \mathrm{SP})$ vs $T_2 = (3, -, \mathrm{SR})$) where $k \neq k_2$. Even if the König lift were proved for the symmetric case, it would not handle the asymmetric pairs that witness the worst case.

Step 3 (proof) is not the right next move

Plan-step 3 from `two_approaches_comparison.tex` was "prove inclusion via transfer matrix / fibre lifting," assuming the candidate partition was empirically correct. The candidate is *not* empirically correct beyond $k = 6$, so trying to prove the wrong statement is futile. Instead the right next move is:

1. Find the right induced $\widetilde{\mathcal{F}_2}$ at $k = 9$: study the 8 surviving triple-partitions, see if they have a common structural description (e.g. via the T_2 annular triangulation).

2. Or abandon the "$\widetilde{\mathcal{F}_2}$ is a partition" framing entirely and look for a different structure on γ that T_2 induces and that suffices for chain pigeonhole.

Reassessment of Approach 2

Approach 2 (König lift) was preferred in `two_approaches_comparison.tex` on the grounds that "the hard step is already proven, only the induced-partition piece is conjectural." These findings show:

- The induced-partition piece is *not* just conjectural — the specific construction in the worst-case note is *wrong* for $k > 6$.
- The König-overlap proposition (when both tires give direct γ-face partitions) is still cleanly proved; it just applies to fewer cases than was hoped.

Updated ranking

Both approaches now have known structural obstacles:

- **Approach 1 (2-SAT, `rainbow_proof.tex`)**: single open conjecture (2-SAT solvability), empirically true for all tested $\sigma \in \mathcal{P}_m$ at $m \in \{4, 6\}$. Limited to $m \in \{4, 6\}$ (SP feasibility) but at least empirically holds throughout that range.
- **Approach 2 (König lift, `worst_case_proof_sketch.tex`)**: König-overlap prop proved, but the natural induced-partition construction is empirically wrong at $k = 9$. Asymmetric pairs (where the worst case actually lives) are not covered at all.

Both approaches give partial structural results. Neither closes the chain-pigeonhole step in its full generality. The honest status: chain pigeonhole has no full proof yet, and both attempted attacks have specific empirical limits.