COLORING NESTED TIRE GRAPHS

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Abstract. We establish the foundational definitions for studying the Four Colour Theorem through nested level-structures on plane triangulations. A *level source* of a triangulation G induces a BFS layering of G, which in turn endows the inner planar dual G' with a *dual depth* grading. We isolate the basic object of study — the *tire graph* T, a plane graph whose outer and inner boundaries bound an annular region triangulated by the *annular edges* E_{ann} — and define its *partial tire dual* $D(T)$, the dual restricted to T's annular faces together with leaves recording the boundary edges.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

The structural study of such a minimal counterexample is the overarching motivation for the present line of work. This first paper establishes the foundational vocabulary — level sources, dual depth, tire graphs, and partial tire duals — on which subsequent papers in the series build. In particular, the companion paper [2] uses these definitions to develop nested-cycle structure theorems and chain-pigeonhole conjectures for tire annular subgraphs of G'.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

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Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \operatorname{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

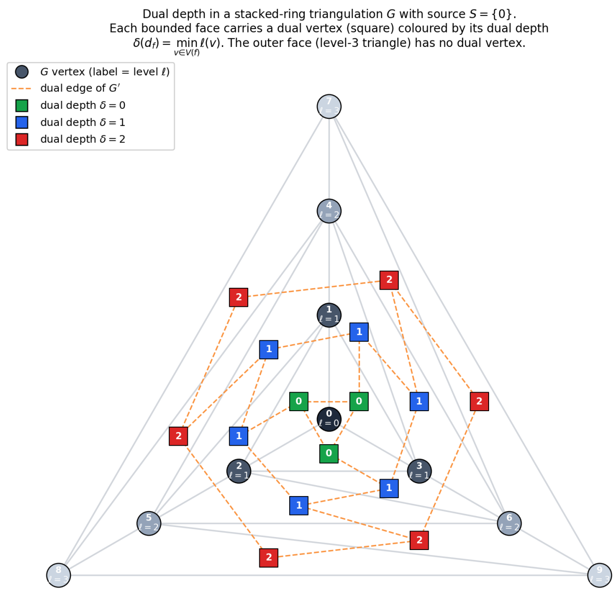

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.5 (Tire graph)**.** A *tire graph* consists of a plane graph T together with an *outer boundary* $B_{\text{out}} \subseteq T$ and an *inner outerplanar graph* $O \subseteq T$ with $V(B_{\text{out}}) \cap V(O) = \emptyset$, where

- B_{out} is either a simple cycle of length ≥ 3 or a single vertex (a *degenerate outer boundary*);
- O is an outerplanar graph; its *inner boundary* B_{in} is the closed walk in O that traces the boundary of O's outer face in the inherited embedding, which is a simple cycle when O is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if $|V(O)| = 1$, we say T has a *degenerate inner boundary*.

At most one of $B_{\text{out}}, B_{\text{in}}$ may be degenerate. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \qquad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T, the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R.

When B_{out} is a simple cycle and O is 2-connected, R is a closed annulus. More generally, R is a closed planar region that may fail to be a 2-manifold at cut-vertices of O (where two "lobes" of the depth-d region meet at a single vertex); the inner boundary B_{in} is then a non-simple closed walk that visits the cut-vertex multiple times. The relaxed definition accommodates outerplanar inner graphs with bridges, cut-vertices, or multiple connected components. When either boundary is degenerate, R is a closed disk with that vertex as apex.

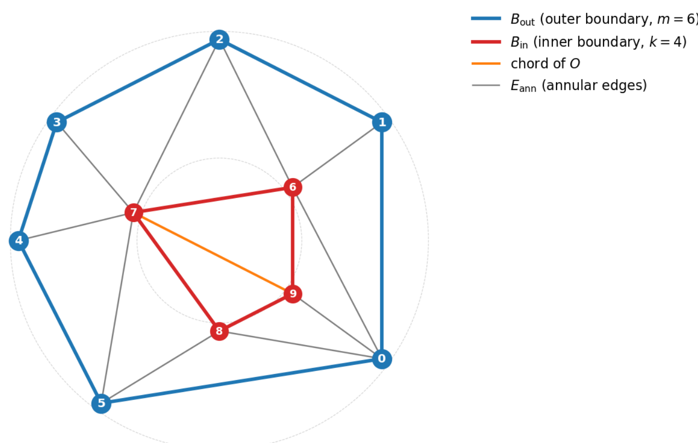

FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \ldots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \ldots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7\text{–}9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Remark 1.6. Let $m = |V(B_{\text{out}})|$ and $k = |V(B_{\text{in}})|$. By Euler's formula on the annular (resp. disk) region R, the tire graph has $m+k$ triangular faces inside R and $|E_{\text{ann}}| = m + k$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(m, k) = 1$), there are $m+k-1$ triangular faces and $|E_{\text{ann}}| = m + k - 1$.

Definition 1.7 (Partial tire dual)**.** Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph in the sense of Definition 1.5, and let F_{ann} denote the set of triangular faces of T in the closed annular region between B_{out} and B_{in}. The *partial tire dual* of T, written $D(T)$, is the graph defined as follows.

Vertices.

(V1) For each face $f \in F_{\text{ann}}$, an *interior vertex* d_f of $D(T)$.
(V2) For each edge $e \in E(B_{\text{out}})$, a *leaf vertex* ℓ_e^{out}.
(V3) For each occurrence of an edge in the closed walk B_{in} (= the outer-face boundary walk of O), a *leaf vertex* ℓ_e^{in}. (When O is 2-connected each edge

appears once; cut-vertices and bridges of O may cause an edge or vertex to appear more than once.)

Edges.

(E1) For each edge $e \in E(T)$ whose two incident faces both lie in F_{ann} (an *interior annular edge*), one edge $\{d_{f_1}, d_{f_2}\} \in E(D(T))$ where $f_1, f_2 \in F_{\mathrm{ann}}$ are the two annular faces incident to e.

(E2) For each $e \in E(B_{\mathrm{out}})$, one edge $\{d_f, \ell_e^{\mathrm{out}}\} \in E(D(T))$ where $f \in F_{\mathrm{ann}}$ is the unique annular face incident to e. The leaf ℓ_e^{out} has degree 1.

(E3) For each occurrence of e on the boundary walk B_{in}, one edge $\{d_f, \ell_e^{\mathrm{in}}\} \in E(D(T))$ where $f \in F_{\mathrm{ann}}$ is the annular face incident to e on the side of that occurrence. The leaf ℓ_e^{in} has degree 1.

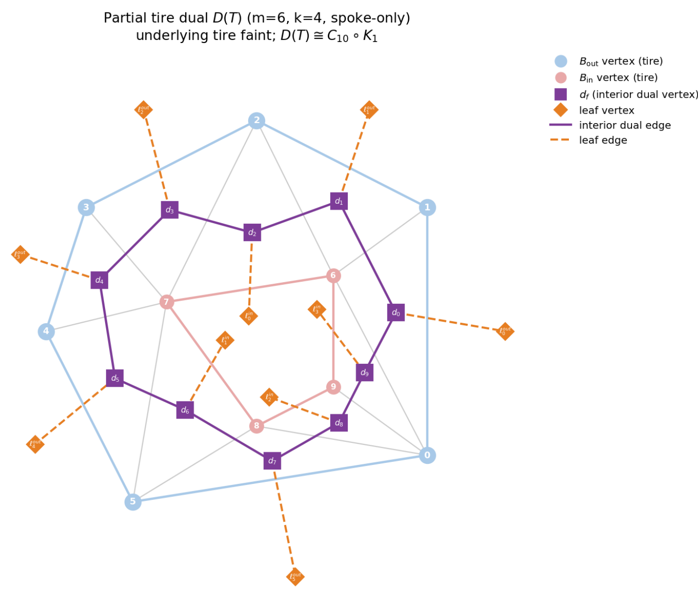

FIGURE 3. The partial tire dual $D(T)$ (purple squares + orange diamonds) drawn on top of a small tire graph T (faint) with $m = 6$ and $k = 4$. The ten interior vertices d_f at the centroids of the annular triangles form a single 10-cycle (solid purple); each boundary edge of the annular region (either of B_{out} or of B_{in}) contributes a degree-1 leaf (orange diamond) attached to the unique annular face incident to it (dashed orange), giving the structure $C_{10} \circ K_1$ analysed in the companion paper [2].

REFERENCES

[1] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.
[2] E. Bauerfeld, *Coloring Nested Tire Dual Graphs*, manuscript (math-research repository), 2026.

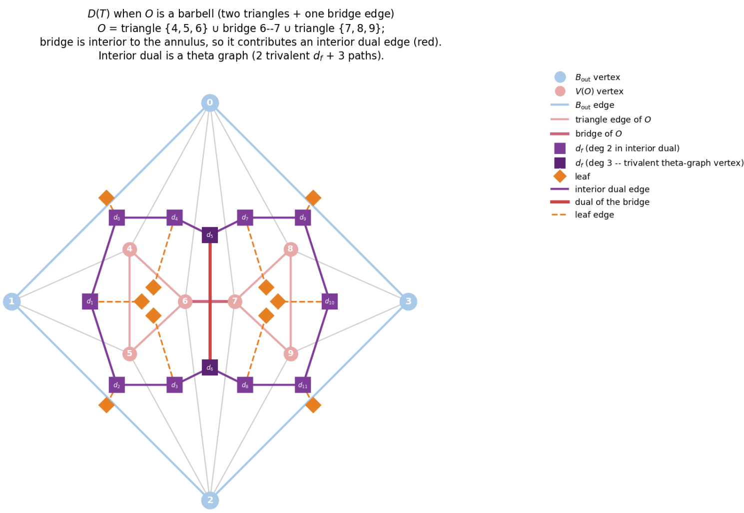

FIGURE 4. Partial tire dual $D(T)$ when the inner outerplanar graph O has a bridge — here a non-trivial edge cut connecting two disjoint triangles. B_{out} is a 4-cycle on $\{0, 1, 2, 3\}$ and O is the barbell: triangle $\{4, 5, 6\}$ together with triangle $\{7, 8, 9\}$ joined by the bridge edge 6–7 (removing the bridge disconnects O). Because both faces incident to the bridge are annular triangles, the bridge contributes an *interior dual edge* (highlighted in red) rather than two leaves; consequently the interior dual subgraph is no longer the single $(n+m)$-cycle of the spoke-only case, but a theta graph: the two trivalent vertices d_5, d_6 (the bridge-incident annular faces) are joined by three internally vertex-disjoint paths in $D(T)$. Leaves come only from B_{out} ($n = 4$ leaves) and the six non-bridge edges of O ($m_\partial = 6$ leaves, three for each triangle).