

# EVEN LEVEL GRAPH GENERATORS

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ABSTRACT.

## 1. INTRODUCTION

## 2. DEFINITIONS

Throughout,  $G = (V, E)$  is a plane maximal planar graph (a triangulation) with a fixed planar embedding  $\Pi_G$ . We write  $|V| = n$ , so  $|E| = 3n - 6$  and  $G$  has  $2n - 4$  triangular faces.

**Definition 2.1** (Level source). A *level source* of  $G$  is any vertex  $v \in V$ ; we write  $S = \{v\}$  for the level-0 source.

**Definition 2.2** (Levels). Given a level source  $S \subseteq V$ , the *level* of  $v \in V$  is  $\ell_G(v) = \text{dist}_G(v, S)$ , the graph distance from  $v$  to the nearest source vertex.

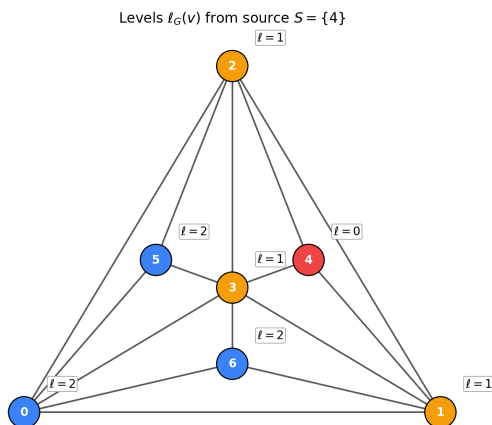


FIGURE 1. BFS levels from the degree-3 vertex source  $S = \{4\}$ . The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

**Definition 2.3** (Level cycle). A *level cycle* of  $G$  (with respect to a level source  $S$ ) is a simple cycle in  $G$  all of whose vertices have the same level.

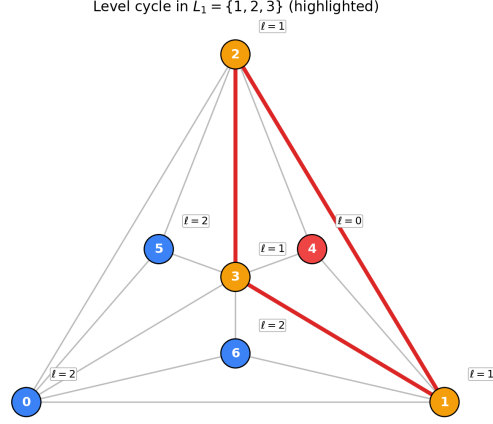


FIGURE 2. A level cycle in the triangulation of Figure 1. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

**Definition 2.4** (Edge switch). Let  $G$  be a triangulation with level source  $S$ , and let  $e = uv$  be an edge of a level cycle of  $G$ . The *edge switch* at  $e$  is the edge flip on  $e$ : writing  $uvw$  and  $uvx$  for the two triangular faces of  $G$  containing  $e$ , the edge  $uv$  is removed and the edge  $wx$  is added. As with any edge flip, the result is a triangulation on the same vertex set provided  $w$  and  $x$  are non-adjacent in  $G$ .

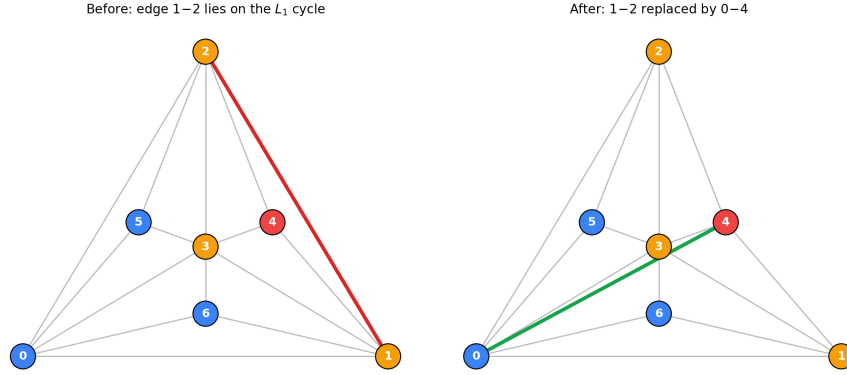


FIGURE 3. An edge switch on the level cycle of Figure 2. The chosen cycle edge 1–2 is shared by the triangular faces  $(0, 1, 2)$  and  $(1, 2, 4)$ ; the switch deletes 1–2 (red, left) and inserts 0–4 (green, right). Vertex colours indicate the original levels in  $G$ .

**Definition 2.5** (Parity subgraph). Let  $G$  be a triangulation with level source  $S$ , and let  $G'$  be a triangulation on the same vertex set as  $G$ . The *even parity subgraph*  $E_{G,S}(G')$  is the subgraph of  $G'$  induced by  $\{v \in V : \ell_G(v) \equiv 0 \pmod{2}\}$ . The *odd parity subgraph* is defined analogously for odd  $\ell_G$ .

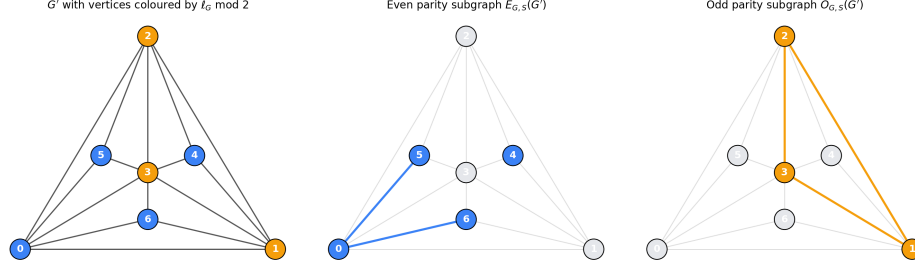


FIGURE 4. Parity subgraphs of  $G' = T$  with respect to the level structure of Figure 1 (here we take  $G = G' = T$ ). Left:  $T$  with vertices coloured by  $\ell_G \bmod 2$  (blue = even, orange = odd). Middle: the even parity subgraph  $E_{G,S}(G')$ , induced on  $\{0, 4, 5, 6\}$ ; only edges with both endpoints even appear. Right: the odd parity subgraph  $O_{G,S}(G')$ , induced on  $\{1, 2, 3\}$ ; the highlighted triangle shows that  $O_{G,S}(G')$  is not bipartite for this choice of  $G'$ .

### 3. OUTERPLANARITY OF LEVEL COMPONENTS

For each integer  $k \geq 0$  and each  $(G, S)$ , write  $L_k$  for the subgraph of  $G$  induced by the level- $k$  vertices. A *level component* of  $G$  (with respect to  $S$ ) is a connected component of some  $L_k$ .

**Theorem 3.1.** *For every plane triangulation  $G$  and every level source  $S$  of  $G$ , every level component of  $G$  is outerplanar.*

*Proof.* Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph  $L_k$  is outerplanar. For  $k = 0$ ,  $L_0 = S$  is a single vertex and is trivially outerplanar.

Fix  $k \geq 1$  and let  $D_k$  be the drawing of  $L_k$  inherited from  $\Pi_G$ . Let  $F^*$  be the face of  $D_k$  containing the source. Suppose for contradiction that some  $u \in L_k$  does not lie on  $\partial F^*$ , so  $u$  lies on the boundary of some other face of  $D_k$ . Take any path  $P$  in  $G$  from  $v_0 \in S$  to  $u$ . As a curve in  $\Pi_G$ ,  $P$  starts in  $F^*$  and ends at a point off  $\partial F^*$ , so it must transition from  $F^*$  to a different face of  $D_k$ ; in a planar embedding this can happen only at a vertex of  $D_k$ , that is, at a level- $k$  vertex  $w$  on  $P$ . Either  $w \neq u$  (so  $P$  has length  $\geq \text{dist}_G(S, w) + 1 \geq k + 1$ ), or  $w = u$  (contradicting  $u \notin \partial F^*$ ). Since every  $S$ -to- $u$  path has length  $\geq k + 1$ ,  $\text{dist}_G(S, u) \geq k + 1$ , contradicting  $u \in L_k$ .  $\square$

### 4. EVEN LEVEL GRAPHS

**Definition 4.1** (Even Level Graph). A plane triangulation  $G$  with level source  $S$  is an *Even Level Graph* if every level cycle of  $G$  has even length.

**Theorem 4.2.** *Every Even Level Graph is 4-colorable.*

*Proof.* Since adjacent vertices in  $G$  have levels differing by at most 1, any edge between two same-parity endpoints in fact connects two vertices at the same level. Hence

$$E_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i}, \quad O_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i+1},$$

and each  $L_k$  is bipartite because its cycles are level cycles of  $G$ , which have even length by hypothesis. Choose a 2-coloring of  $E_{G,S}(G)$  in  $\{\text{red}, \text{blue}\}$  and a 2-coloring of  $O_{G,S}(G)$  in  $\{\text{yellow}, \text{green}\}$ . Same-parity edges of  $G$  are properly colored by the respective bipartition; opposite-parity edges connect  $\{\text{red}, \text{blue}\}$  to  $\{\text{yellow}, \text{green}\}$ . The combined assignment is a proper 4-coloring of  $G$ .  $\square$

**Definition 4.3** (Derived level graph). Let  $G$  be an Even Level Graph with level source  $S$ , and let  $E$  and  $O$  denote the edge sets of the even and odd parity subgraphs  $E_{G,S}(G)$  and  $O_{G,S}(G)$ . A *derived level graph* of  $G$  is a triangulation  $G'$  on the same vertex set as  $G$  obtained by a sequence of edge switches (Definition 2.4), each acting on an edge of  $E$  or of  $O$ . We do not update  $E$  or  $O$  to reflect the level structure of intermediate triangulations: throughout the sequence, an edge is classified as belonging to  $E$  (resp.  $O$ ) if and only if both of its endpoints have even (resp. odd) level in  $G$ .

A derived level graph  $G'$  is *valid* if both  $E_{G,S}(G')$  and  $O_{G,S}(G')$  contain only even cycles.

**Definition 4.4** (Bridge switch). Let  $G'$  be a triangulation reached from an Even Level Graph  $G$ , with parity classes inherited from  $G$  as in Definition 4.3. An edge switch on an edge  $e \in E \cup O$  of  $G'$ , replacing  $uvw, uvx$  by the edge  $wx$ , is a *bridge switch* if either

- the new edge  $wx$  is a cross-parity edge (one endpoint even, the other odd), so  $wx$  enters neither parity subgraph; or
- $wx$  is a same-parity edge and is a *bridge* in the parity subgraph it joins – that is,  $w$  and  $x$  lie in different connected components of that parity subgraph, so adding  $wx$  creates no new cycle.

**Definition 4.5** (Bridge-derived level graph). A *bridge-derived level graph* of an Even Level Graph  $G$  is a triangulation obtained from  $G$  by a sequence of bridge switches (Definition 4.4).

Because a bridge switch never closes a cycle in a parity subgraph, it never introduces an odd cycle there. As an Even Level Graph has bipartite parity subgraphs (every level cycle is even), every bridge-derived level graph has bipartite parity subgraphs as well, and so is automatically a valid derived level graph. Equivalently, the first Betti number of each parity subgraph is non-increasing along any sequence of bridge switches.

**Definition 4.6** (Intertwining tree). A maximal planar graph  $G$  is an *intertwining tree* if its vertex set can be partitioned into two sets  $A$  and  $B$  such that both induced subgraphs  $G[A]$  and  $G[B]$  are trees.

**Theorem 4.7.** *A maximal planar graph  $G$  is an intertwining tree if and only if its dual  $G^*$  has a Hamiltonian cycle.*

*Proof.* ( $\Rightarrow$ ) Let  $V(G) = A \sqcup B$  with  $G[A]$  and  $G[B]$  trees. Every triangular face  $\{x, y, z\}$  of  $G$  meets both  $A$  and  $B$ : if all three vertices were in  $A$  the triangle would be a cycle in the tree  $G[A]$ , and likewise for  $B$ . Draw a closed curve through the faces of  $G$  separating the  $A$ -vertices from the  $B$ -vertices within each face. Since every face is split, the curve visits every face exactly once and crosses an edge of  $G$  precisely when that edge joins  $A$  to  $B$ ; it is therefore a Hamiltonian cycle of  $G^*$ .

( $\Leftarrow$ ) Let  $H$  be a Hamiltonian cycle of  $G^*$ . Drawn in the plane,  $H$  is a Jordan curve visiting every face of  $G$  once; let  $A$  and  $B$  be the vertices of  $G$  interior and exterior to  $H$ . The  $2n - 4$  edges of  $H$  cross exactly the edges of  $G$  between  $A$  and  $B$ , leaving  $(3n - 6) - (2n - 4) = n - 2$  edges inside  $G[A]$  and  $G[B]$  together. The edges inside  $A$  lie in the disk bounded by  $H$  and span  $A$  without enclosing a face (each face is cut by  $H$ ), so  $G[A]$  is a tree; likewise  $G[B]$ .  $\square$

**Conjecture 4.8.** Every maximal planar graph is a bridge-derived level graph of some Even Level Graph, an intertwining tree, or both.

Since a bridge-derived level graph is automatically a valid derived level graph, this is a stronger statement than the corresponding conjecture phrased with arbitrary  $E/O$  switches; it is also the form that the evidence below actually supports.

By Theorem 4.7, the intertwining-tree disjunct fails for  $G$  exactly when  $G^*$  is a counterexample to Tait’s conjecture. The smallest such  $G^*$  have 38 vertices (Holton–McKay [1], exactly 6 graphs), so the smallest triangulations that are not intertwining trees occur at  $n = 21$  and there are exactly 6 of them. Below  $n = 21$  every maximal planar graph is an intertwining tree, which is why the disjunction holds trivially in that range.

**Empirical status.** For each isomorphism class of maximal planar graphs on  $n$  vertices, we ask whether (i) some isomorphic representative is a bridge-derived level graph of some Even Level Graph, and/or (ii) it is an intertwining tree. The conjecture holds for the class iff at least one of (i), (ii) holds. Below  $n = 21$  condition (ii) holds for *every* class, so the table mainly records how far the bridge-derived disjunct (i) reaches on its own. We classified bridge-derivability exhaustively for  $n \leq 9$ , where every backward bridge-orbit can be enumerated in full.

$n$	# iso	bridge only	inter. only	both	missing	status
6	2	0	0	2	0	holds
7	5	0	1	4	0	holds
8	14	0	2	12	0	holds
9	50	0	14	36	0	holds

Here “bridge only” counts classes that are bridge-derived but not intertwining trees, “inter. only” the reverse, and “both” the intersection; “missing” counts classes that are neither (a counterexample). The “bridge only” column is 0 throughout this range precisely because every class is an intertwining tree for  $n \leq 20$ ; the “inter. only” counts (1, 2, 14) are the classes that the bridge-derived disjunct alone does not yet reach, showing that bridge-derivability is strictly weaker than “intertwining tree” here and that the two disjuncts genuinely complement one another.

**The boundary case  $n = 21$ .** The first triangulations that are *not* intertwining trees are the six duals of the Holton–McKay graphs, at  $n = 21$ . For the disjunction to survive at  $n = 21$ , each of these six must be a valid derived level graph. We find:

- All six duals are confirmed not intertwining trees (exhaustive check of all  $2^{20} - 1$  vertex bipartitions), consistent with Theorem 4.7.
- Two of the six are themselves Even Level Graphs (for a suitable source vertex), hence trivially valid derived level graphs. So the disjunction holds for them through the derived-level-graph disjunct – the first instances where that disjunct does work the intertwining-tree disjunct cannot.

- The remaining four are not Even Level Graphs for any source, and their full  $E/O$ -orbits ( $\sim 10^8$  states per source labelling) are far too large to exhaust. Restricting to *bridge switches* (Definition 4.4) shrinks the relevant orbits by roughly two orders of magnitude and, crucially, keeps every reachable triangulation valid. A backward bridge-switch search over the valid parity partitions found an Even Level Graph witness for each of the four, so all four are *bridge-derived level graphs* (Definition 4.5) and hence valid derived level graphs. The witnessing orbits are small – between a few hundred and  $\sim 1.7 \times 10^5$  states – even though other parity partitions of the same triangulations have orbits exceeding  $10^6$ ; finding one good partition suffices. Each witness is in fact only a *handful* of bridge switches from its dual: the explicit Even Level Graph, parity labelling, and bridge-switch sequence are recorded for all six – path lengths 3, 1, 2, 4 for these four and 0 for the two that are Even Level Graphs outright – and each step has been verified to be a valid bridge switch.

Thus at  $n = 21$  the disjunction is confirmed for all six critical iso classes: two are Even Level Graphs outright, and the other four are bridge-derived level graphs. The bridge-switch restriction is what made the search tractable – it both shrinks the orbit and guarantees validity, so any Even Level Graph located in a backward orbit is an immediate witness.

#### REFERENCES

- [1] D. A. Holton and B. D. McKay. *The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices*. Journal of Combinatorial Theory, Series B, 45(3):305–319, 1988.