Towards a proof of the antipodal-chord rainbow theorem (corrected statement with sharp threshold)

Statement and status

Theorem 1 (Antipodal-chord rainbow theorem, claimed). *Let $T = T(m_1, m)$ be a tire whose outer boundary B_{out} has length m_1, whose inner boundary B_{in} has even length $m \in \{4, 6\}$, and whose inner outerplanar graph is $O = B_{\text{in}} \cup \{(v_0, v_{m/2})\}$ (the antipodal chord). Assume the Steiner-poor model and the balanced annular triangulation.*

If $m_1 \geq m - 1$, then

$$\pi_D(\mathcal{C}(T)) \;=\; \mathcal{P}_m \;:=\; \big\{\sigma \in \{1,2,3\}^m : \;(\sigma_0, \ldots, \sigma_{m/2-1}),\; (\sigma_{m/2}, \ldots, \sigma_{m-1}) \in \mathrm{Perm}(\{1,2,3\}^{m/2})\big\}.$$

In particular π_D contains the S_3-orbit of the rainbow configuration $(a, b, c, b, c, \ldots, b, c, a)$.

The threshold $m_1 \geq m - 1$ is sharp: at $m_1 = m - 2$ (when $m = 6$) one has $|\pi_D| = 18$ and the rainbow is missing.

Status. The "$\subseteq$" half is fully proven below. The "$\supseteq$" half is partially proven — reduced to a cyclic 2-SAT solvability statement which is empirically true for every $\sigma \in \mathcal{P}_m$ (between 6 and 18 satisfying orientation assignments per σ) but for which a clean general proof is deferred. Sharpness is confirmed by explicit forcing counterexamples.

The easy direction: $\pi_D \subseteq \mathcal{P}_m$

Lemma 2 (O-face dual constraint). *For any $\sigma \in \pi_D(\mathcal{C}(T))$, the restrictions of σ to the two halves $\{0, \ldots, m/2 - 1\}$ and $\{m/2, \ldots, m - 1\}$ are each permutations of $\{1,2,3\}^{m/2}$ with $m/2$ distinct entries (so $\in \mathrm{Perm}$ when $m/2 = 3$).*

Proof. The antipodal chord splits O into two faces F_A, F_B. Under SP, F_A has boundary B_{in}-edges $e_0, \ldots, e_{m/2-1}$ and F_B has $e_{m/2}, \ldots, e_{m-1}$. In $T'_{f'}$ each face dual u_{F_A} has degree exactly $m/2$, with one D-spoke per B_{in}-edge. Proper edge 3-coloring at u_{F_A} requires all $m/2$ incident spokes to have distinct colors; since we only have 3 colors and $m/2 \in \{2, 3\}$, this is exactly the "$m/2$ distinct entries among $\{1,2,3\}$" condition. Symmetric for F_B. □

Consequently $|\pi_D| \leq |\mathcal{P}_m| = 36$ at both $m = 4$ and $m = 6$.

Reduction of $\mathcal{P}_m \subseteq \pi_D$ to a cyclic 2-SAT

We work in the balanced annular triangulation with D-positions $p_j = \lfloor j(m_1 + m)/m \rfloor$ on $T'_{\text{ann}} = C_n$, $n := m_1 + m$.

Setup of orientations and gaps

At each D-position p_j, the two incident cycle edges (e_{p_j-1}, e_{p_j}) are constrained by Lemma 2's consequence at the cycle vertex p_j to satisfy $\{c(e_{p_j-1}), c(e_{p_j})\} = \{1,2,3\} \setminus \{\sigma_j\} =: \{\alpha_j, \beta_j\}$ with $\alpha_j < \beta_j$. Let $o_j \in \{0,1\}$ encode the choice:

$$o_j = 0 \;\Leftrightarrow\; c(e_{p_j-1}) = \alpha_j,\; c(e_{p_j}) = \beta_j; \qquad o_j = 1 \;\Leftrightarrow\; \text{swapped.}$$

Between consecutive D-positions p_j, p_{j+1} (cyclically) sit $L_j - 1 \geq 0$ U-positions, where $L_j := p_{j+1} - p_j$ is the gap arc length. Cycle edges within the gap form a path; the two endpoints have already-fixed colors $R_j := c(e_{p_j})$ (right side of p_j) and $L_{j+1} := c(e_{p_{j+1}-1})$ (left side of p_{j+1}), both determined by o_j and o_{j+1} via $R_j = \beta_j$ if $o_j = 0$ else α_j, and $L_{j+1} = \alpha_{j+1}$ if $o_{j+1} = 0$ else β_{j+1}.

Lemma 3 (Gap feasibility)**.** *Given R_j, L_{j+1} and gap arc length $L_j \geq 1$, the gap admits a proper edge 3-coloring of its interior cycle edges and U-spokes iff*

$$L_j = 1 \;\Rightarrow\; R_j = L_{j+1}; \qquad L_j = 2 \;\Rightarrow\; R_j \neq L_{j+1}; \qquad L_j \geq 3 \;\Rightarrow\; \textit{no constraint.}$$

For $L_j \geq 3$ at least one extension exists (chromatic-polynomial nonzero).

Proof. $L_j = 1$: a unique cycle edge equals both R_j and L_{j+1} (it *is* the edge connecting them). $L_j = 2$: two adjacent edges with fixed endpoints R_j, L_{j+1} require $R_j \neq L_{j+1}$ for proper coloring at the U-position between. $L_j \geq 3$: straightforward to construct any pair (R_j, L_{j+1}) by alternating colors with at least one "flexible" cycle edge between. □

Reduction to cyclic 2-SAT

When $m_1 \geq m$, every $L_j \geq 2$; when $m_1 = m - 1$, exactly one $L_j = 1$ (with the rest $L_j = 2$).

Proposition 4 (2-SAT reduction)**.** *For $\sigma \in \mathcal{P}_m$, the existence of a proper edge 3-coloring of $T'_{f'}$ inducing σ on D-spokes is equivalent to the existence of $(o_0, \ldots, o_{m-1}) \in \{0,1\}^m$ satisfying, for each j:*

- *if $L_j = 1$: $R_j(o_j) = L_{j+1}(o_{j+1})$ (an equality clause);*
- *if $L_j = 2$: $R_j(o_j) \neq L_{j+1}(o_{j+1})$ (an inequality clause);*
- *if $L_j \geq 3$: no constraint.*

Each clause is a 2-CNF clause on the variables o_j, o_{j+1}, so the system is a cyclic 2-SAT.

Proof. Cycle edges within a gap depend only on R_j and L_{j+1} (by Lemma 3), which depend only on (o_j, o_{j+1}). The proper-coloring constraints at the U-positions and D-positions not at the boundary of the gap are taken care of by gap feasibility. □

The remaining step

Conjecture 5 (2-SAT solvability)**.** *For every $\sigma \in \mathcal{P}_m$ with $m \in \{4, 6\}$ and the balanced triangulation with $m_1 \geq m - 1$, the cyclic 2-SAT system of Proposition 4 is satisfiable.*

Empirical verification. For $m = 6$, $m_1 \in \{5, \dots, 8\}$ all 36 elements of $\mathcal{P}_6$ admit between 6 and 18 satisfying orientation assignments (script: `experiments/orbit_decomposition.py`). No counterexample exists in the tested range.

Why a clean general proof is non-trivial. The straightforward attempt — "$o_j = 0$ for all j is always satisfying" — fails: the rainbow $\sigma = (1, 2, 3, 2, 3, 1)$ at $m_1 = 6$ has $R_4(0) = L_5(0) = 2$, violating the $L_4 = 2$ clause. A satisfying assignment exists (e.g. $(0, 0, 0, 0, 1, 0)$) but the needed flip depends on σ. A correct general proof must analyse the implication graph of the cyclic 2-SAT, which has σ-dependent forbidden combinations.

The forbidden (o_j, o_{j+1}) combo at a length-2 gap depends on the pair (σ_j, σ_{j+1}) via the third color $u := \{1, 2, 3\} \setminus \{\sigma_j, \sigma_{j+1}\}$:

case	forbidden (o_j, o_{j+1})	implication
$\sigma_j = \sigma_{j+1}$	$(0, 1), (1, 0)$	$o_j = o_{j+1}$
$\sigma_j, \sigma_{j+1} \in \{2, 3\}$, $u = 1$	$(1, 0)$	$o_j = 1 \Rightarrow o_{j+1} = 1$
$(\sigma_j, \sigma_{j+1}) = (1, 3)$, $u = 2$	$(1, 1)$	$o_j = 1 \Rightarrow o_{j+1} = 0$
$(\sigma_j, \sigma_{j+1}) = (3, 1)$, $u = 2$	$(0, 0)$	$o_j = 0 \Rightarrow o_{j+1} = 1$
$\sigma_j, \sigma_{j+1} \in \{1, 2\}$, $u = 3$	$(0, 1)$	$o_j = 0 \Rightarrow o_{j+1} = 0$

A clean proof of Conjecture 5 likely proceeds by: (i) Showing the implication graph on $\{(o_j, o_{j+1})\}$ is bipartite in a strong sense that prevents the cyclic chain from forcing a contradiction; (ii) Reducing modulo color symmetry (S_3 acts on σ and on the orientation labels jointly), since $\mathcal{P}_m$ has $|\mathcal{P}_m|/6 = 6$ orbits and verifying solvability on 6 representatives suffices.

The sharpness counterexample

Proposition 6 (Sharpness at $m_1 = m - 2$)**.** *At $m = 6, m_1 = 4$, the rainbow $\sigma = (1, 2, 3, 2, 3, 1)$ is* not *in π_D.*

Proof. Balanced triangulation gives two length-1 gaps, between D-positions $(2, 3)$ (with $(\sigma_1, \sigma_2) = (2, 3)$, forcing $c(e_2) = 1$) and $(7, 8)$ (with $(\sigma_4, \sigma_5) = (3, 1)$, forcing $c(e_7) = 2$). Propagating through T'_{ann} via the remaining length-2 gaps: $c(e_3) = 2$ (from D-position 3), $c(e_8) = 3$ (from D-position 8), $c(e_9) = 2$ (from U-position 9 with $c(e_8) = 3$), $c(e_0) = 3$ (from D-position 0), $c(e_1) = 1$ (from U-position 1 with $c(e_0) = 3$), but the constraint at D-position 2 then demands $c(e_1) = 3 \neq 1$. Contradiction. □

What this gives us

Combining Lemma 2, Proposition 4, and the (empirically verified, not proven) Conjecture 5:

Corollary 7 (Provisional)**.** *For $m \in \{4, 6\}$ and balanced triangulation with $m_1 \geq m-1$: $\pi_D = \mathcal{P}_m$, in particular the rainbow S_3-orbit lies in π_D.*

Corollary 8 (Chain pigeonhole simplification, conditional)**.** *Conditional on Conjecture 5, the chain-pigeonhole step at $|\gamma| = m$ for an antipodal-chord SP tire T_1 with $m_1 \geq m - 1$ reduces to: $\pi_U(\mathcal{C}(T_2)) \cap \mathcal{P}_m \neq \emptyset$, i.e. a non-empty intersection on the perms-per-half set rather than on all of $\{1, 2, 3\}^m$.*

What's missing

A clean structural proof of Conjecture 5. Two candidate routes:

1. **S_3-equivariant case analysis.** Reduce to 6 representative σ's (one per S_3-orbit) and explicitly construct a satisfying o for each at $m_1 = m - 1$ (the tightest case).

2. **Global implication-graph analysis.** Show that the cyclic chain of implications never produces a contradiction. This likely involves a parity argument on the number of "odd implications" around the cycle.