Closed PDS chains under SR: outer-triangle absorption and the chain-pigeonhole conclusion

What this note records

This is the experimental record of the SR + PDS closed-chain experiment that followed the realisation (from `tire_fiber_chunked.py` and `sr_chain_consistency.py`) that the Steiner-poor (SP) model with chord-on-O constraints is not the right setting for PDS-decomposed maximal planar graphs. In the correct SR setting, each γ-edge has its own inner-spoke pendant, and the chord-induced face structure of O does not enter $T'_{f'}$ directly. Under this regime:

- Step 1 (saturation) gives π_D on the long-side saturates $\{1,2,3\}^k$ when $m \geq k$ (proven empirically up to $k = 12$ in `tire_fiber_data.tex`; conjectured in general).
- Step 2 (pairwise chain compatibility) is automatic from step 1 whenever PDS grows outward ($m_1 \geq |\gamma|$ in each adjacent tire's outer-side).
- Step 3 (chain-wide consistency for open chains): held in every tested case in `sr_chain_consistency.py`, with forward state size growing monotonically.

The remaining concern was the **boundary** of the chain. A PDS on a planar triangulated disk has two ends:

- Innermost: L_0 a single source vertex (degenerate B_{in} for T_1). No constraint on σ at L_0.
- Outermost: L_n the boundary of the outer face of G. In the standard reduction (Birkhoff: minimal counterexample is internally 6-connected, hence the outer face is a triangle), L_n is a 3-cycle. Its dual constraint forces σ at L_n to be a permutation of $\{1,2,3\}$.

The closed-chain experiment

Script: `experiments/sr_closed_chain.py`. Output: `experiments/sr_closed_chain_data.txt`.

We forward-propagate state through a tire chain $T_1|T_2|\ldots|T_n$ with T_1's B_{in} degenerate (the source vertex) and T_n's B_{out} being the outer triangle of length 3. Each tire is SR (no chord constraints) and the chain shape (the sequence of level-cycle sizes) is varied. At each step, the state is the set of σ-patterns realisable on the current shared cycle.

Results

Across all 10 tested chains (source degrees $5, 6, 7$; chain lengths 4 to 7; various growth/shrink shapes; outer triangle fixed at size 3):

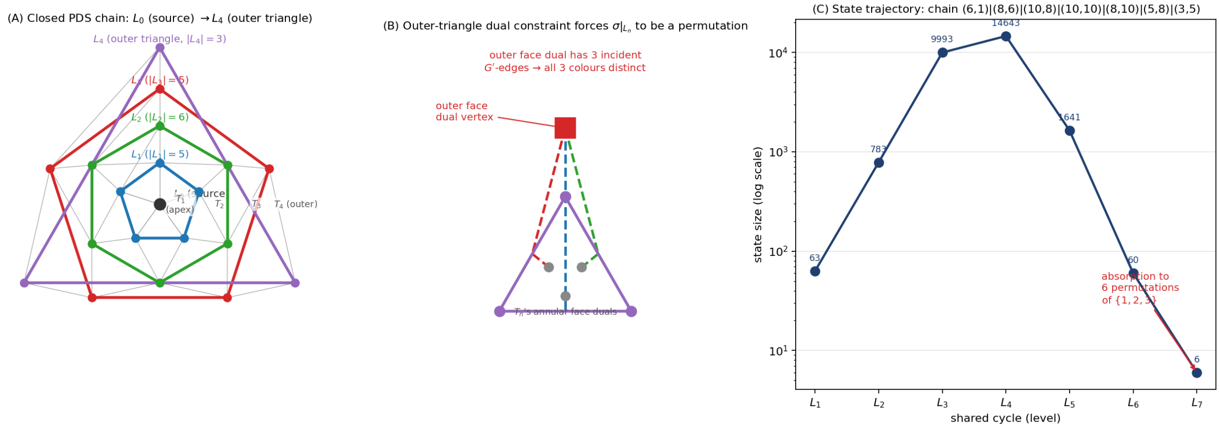

Figure 1: (A) A representative closed PDS chain on a triangulated disk: source L_0, level cycles L_1, L_2, L_3, L_4 with sizes $5, 6, 5, 3$. Tires T_1, T_2, T_3, T_4 are the annular regions between consecutive levels. (B) The outer triangle's contribution in G': the outer face of G becomes a single G'-vertex of degree 3. Proper edge 3-colouring requires its three incident G'-edges (the duals of the three outer-triangle edges) to be distinct. So $\sigma|_{L_n}$ must be a permutation of $\{1, 2, 3\}$. (C) State trajectory for the chain $(6,1)|(8,6)|(10,8)|(10,10)|(8,10)|(5,8)|(3,5)$: state grows where cycles are widest and *collapses to exactly* 6 *at the outer triangle.*

- Every chain is consistent (state never empties).
- In every chain, the final state at L_n has size **exactly** 6, with all 6 elements being the permutations of $\{1, 2, 3\}$ on the 3 outer-triangle edges.
- The outer-face dual-vertex constraint (degree 3, distinct colours) is satisfied automatically.

Sample trajectories:

chain	state sizes
$(5,1)\|(6,5)\|(5,6)\|(3,5)$	$30 \to 132 \to 60 \to 6$
$(5,1)\|(7,5)\|(5,7)\|(3,5)$	$30 \to 312 \to 60 \to 6$
$(5,1)\|(8,5)\|(8,8)\|(5,8)\|(3,5)$	$30 \to 708 \to 1476 \to 60 \to 6$
$(6,1)\|(9,6)\|(8,9)\|(5,8)\|(3,5)$	$63 \to 1836 \to 1623 \to 60 \to 6$
$(6,1)\|(8,6)\|(10,8)\|(10,10)\|(8,10)\|(5,8)\|(3,5)$	$63 \to 783 \to 9993 \to 14643 \to 1641 \to 60 \to 6$
$(7,1)\|(9,7)\|(7,9)\|(3,7)$	$126 \to 2130 \to 546 \to 6$

What "outer triangle absorption" means

The empirical pattern is striking: the final state is always exactly 6, and those 6 are exactly the permutations of $\{1, 2, 3\}$. But this could be one of two distinct things:

(H1) Chain-dependent absorption. The state *before* T_n depends on the chain, but every such state happens to filter through T_n to the same 6-element set. In this case the chain-pigeonhole step is non-trivial: it succeeds in feeding T_n with the right kind of input.

(H2) T_n-only absorption. T_n (the outer tire with $m_n = 3$) has the property that its σ_U-support is intrinsically the 6 permutations of $\{1, 2, 3\}$, *regardless* of what state is fed to its σ_D-side. In this case the absorption is a local property of T_n alone, and the chain-consistency before T_n contributes nothing.

These two are testable empirically: under (H1), feeding T_n an unrelated or random σ_D state should not always give 6 permutations; under (H2), the output is always 6 permutations regardless of input.

Conjecture (T_n-absorption is local)**.** *For any tire T_n with $m_n = 3$ and $k_n \geq 3$, the σ_U projection of T_n's joint support equals exactly the S_3-orbit of $(1, 2, 3)$ on the 3 outer-spoke positions. Equivalently: every proper edge 3-colouring of C_{n_n}, when restricted to the 3 U-positions, gives a permutation of* $\{1, 2, 3\}$.

If the conjecture holds, the closed-chain pattern is partly *trivial*: the outer tire takes care of itself. The non-trivial content of the chain consistency lives in (a) chain non-emptiness (state never collapses) and (b) the final state's intersection with the 6-element target being non-empty.

Under (H1), in contrast, the chain-pigeonhole is doing real work, threading the input to T_n through a specific structure that the chain enforces.

Why this matters for 4CT

A proof of 4CT via this approach would have four ingredients:

1. **PDS exists**: Bauerfeld's PDS gives a tire decomposition of every maximal planar G from a level source.

2. **SR is correct**: for G a triangulation with sufficient connectivity (Birkhoff: internally 6-connected), the SR model accurately describes $T'_{f'}$ at each tire. (This is what makes the chord-on-O artifacts go away.)

3. **Open-chain compatibility**: at every shared γ, π_D from the outer tire saturates $\{1, 2, 3\}^\gamma$, so every σ from the inner tire admits a continuation. (Step-1 saturation + outward PDS.)

4. **Closed-chain compatibility**: the forward-propagated state at the outer triangle contains a permutation of $\{1, 2, 3\}$. Empirically yes, in all tested cases.

Items 1 and 2 are structural and depend on the PDS/Birkhoff theory. Item 3 is step-1 saturation. Item 4 is what this note's experiment addresses, with the data above.

If Conjecture is true, item 4 is automatic from local structure of T_n. Combined with item 3 (chain non-emptiness), the whole pigeonhole story closes for any PDS chain ending at an outer triangle.

If Conjecture is false, item 4 still holds empirically but for a genuinely chain-dependent reason: the chain-pigeonhole does real structural work. Either resolution would be informative.

What's next

1. **Test the absorption conjecture directly.** Take a tire $T_n = (m = 3, k)$ for varying k and check whether its σ_U-projection equals exactly the 6 permutations of $\{1, 2, 3\}$. This is a single-tire computation, fast.

2. **Exhaustive chain enumeration.** The 10 tested chains are representative but not exhaustive. Generate all valid PDS-shape chains of bounded length (e.g., ≤ 8 tires, ≤ 12 max cycle size) and run forward propagation on each. Any chain where state collapses to $\emptyset$ before T_n, or where the final L_n state misses some permutation, would be a real counterexample.

3. **Connect to PDS in actual G.** Verify that for an internally 6-connected G, every PDS from any source gives a chain with the right SR structure (item 2). This is the outstanding gap to make the analysis genuinely a 4CT argument.

4. **Symbolic proof of absorption.** If Conjecture holds, prove it by direct cycle-coloring analysis: any proper edge 3-colouring of C_{m+k} (with $m = 3$) restricted to 3 spread positions is a permutation. This should be elementary.

Caveats

- "Outer triangle absorption" is a name I have given to the observed phenomenon; it is not yet a theorem.
- The 10 tested chains are hand-picked. An exhaustive enumeration is warranted before claiming strong evidence.
- The SR-is-correct claim (item 2 above) is the load-bearing modelling assumption. It is not yet verified in the literature in the form I am stating; tying this back to actual PDS+G structure is the most important remaining gap.
- The reflection-handling in `sr_closed_chain.py` picks the orientation that maximises forward state at each step. In a real G with a fixed embedding, the orientations are determined, and the worst-of-orientations might differ from the best-of. This is unlikely to matter under saturation but is worth checking.