S_3-orbit decomposition of $S_1 \cap S_2$
Do structurally different tires share the same canonical orbits?

The question

Step 2 (`tire_fiber_step2.tex`) reported $|S_1 \cap S_2|$ for 23 adjacent-tire pairs and found all 23 compatible. A follow-up question is whether structurally different (T_1, T_2) pairs that share a cycle-length $|\gamma| = k$ have intersections with the *same orbit structure* — i.e. whether a canonical pattern like the rainbow (a, b, c, b, c, a) persists when we vary T_1 and T_2, or whether each pair gives its own pair-specific orbit.

Script: `experiments/orbit_decomposition.py`; raw output: `experiments/orbit_decomposition_data.txt`.

S_3-closure: a structural sanity check

Permuting the three colors is a symmetry of proper edge 3-coloring, so for any fixed tire T, both S_1 and S_2 must be closed under the diagonal S_3 action on $\{1, 2, 3\}^k$. Hence so is $S_1 \cap S_2$.

Observation (S_3-closure). *In every one of the* 23 *pairs,* $S_1 \cap S_2$ *is closed under the* S_3 *color action. This is structural rather than coincidental.*

Orbit size distribution

For $\sigma \in \{1, 2, 3\}^k$ with 3 distinct color-values, the S_3 orbit has size exactly 6. For σ using 2 distinct color values, the orbit has size 6 (since S_3 acts on the $\binom{3}{2} \cdot 2$ ways of placing them). For σ constant (one color), the orbit has size 3.

Observation (Uniform orbit sizes). *Across all* 23 *pairs, every* S_3*-orbit in* $S_1 \cap S_2$ *has size exactly* 6*, with one single exception: the constant orbit* $\{(1, \ldots, 1), (2, \ldots, 2), (3, \ldots, 3)\}$ *of size* 3*, which appears only in the case* $\gamma = 4$*,* $T_1 = T_2 = (4, -, \mathrm{SR})$*.*

So intersection sizes are essentially $6 \cdot$ (number of orbits) in all but one case.

The rainbow orbit reappears across structurally different pairs

Combining S_3 color action with cyclic rotation of γ gives a larger symmetry group; the combined orbit of $(1, 2, 3, 2, 3, 1)$ has size 36. This single combined orbit appears in three different (T_1, T_2) pairs at $\gamma = 6$:

γ	T_1	T_2
6	$(6, (0, 3), \mathrm{SP})$	$(6, (0, 3), \mathrm{SP})$
6	$(6, (0, 3), \mathrm{SP})$	$(6, (0, 2)(3, 5), \mathrm{SP})$
6	$(6, (0, 3), \mathrm{SP})$	$(3, -, \mathrm{SR})$

All three have $T_1 = (6,(0,3),\mathrm{SP})$ — the side with the *antipodal* chord. The three different T_2 structures range from another antipodal-chord SP tire to a chord-less SP tire to a chordless SR tire. In every case the rainbow orbit is part of (or all of) the intersection.

Observation (Rainbow orbit is T_1-driven at $\gamma = 6$)**.** *At $\gamma = 6$, the rainbow combined orbit (a,b,c,b,c,a) appears* iff *$T_1 = (6,(0,3),\mathrm{SP})$ — the antipodal-chord SP tire. The two-chord SP tire $T_1 = (6,(0,2)(3,5),\mathrm{SP})$ yields different orbits, never the rainbow. So the rainbow pattern is associated with antipodal O-chord topology rather than with the pair (T_1, T_2) as a whole.*

Most-shared canonical orbits

Other canonical combined orbits appear across many structurally different pairs. Top entries:

$\lvert\gamma\rvert$	canonical rep	# distinct pairs	combined-orbit size
4	$(1,2,1,3)$	7	12
4	$(1,2,1,2)$	6	6
4	$(1,2,2,1)$	5	12
4	$(1,2,2,3)$	5	24
3	$(1,2,3)$	4	6
5	$(1,2,1,2,3)$	3	30
6	$(1,2,3,2,3,1)$ (rainbow)	3	36

Observation (Universal orbits dominate)**.** *At each cycle length k, the most-shared canonical orbit appears in roughly 30–40% of tested pairs at that k. These "universal" orbits are typically of the form (a,b,a,c) or (a,b,a,b) at $\gamma = 4$, (a,b,c) at $\gamma = 3$, etc. They survive across chord placements and across SR/SP model choices.*

What this means for chain pigeonhole

If $S_1 \cap S_2$ is always S_3-closed (Obs.) and always contains at least one full S_3-orbit (which, given size 6 for non-constant orbits, is essentially the same as saying $|S_1 \cap S_2| > 0$), then *chain pigeonhole at a single shared cycle is not just a counting fact but a structural one*: the intersection respects color symmetry, never collapses to a thin exotic subset, and contains canonical orbits that recur across structurally varied pairs.

This is a real upgrade on the step-2 data:

- Step 2 only reported $|S_1 \cap S_2| > 0$; it could in principle be a single weird configuration with no symmetry.
- The orbit decomposition shows the intersections always have the full S_3-symmetric shape, with at least one orbit of size 6 (or 3 in the trivial case).
- Several canonical orbits recur across structurally different pairs — evidence that chain compatibility is detecting something structural about γ and the tire-pair topology, not a coincidence of one specific configuration.

Conjecture suggested by the data

Observation (Antipodal-chord rainbow conjecture — *refuted as originally stated*; refined below)**.** Original (incorrect) statement. *Let $T = (m, (0, m/2), \mathrm{SP})$ be a Steiner-poor tire whose inner outerplanar graph O is a cycle of length m together with a single antipodal chord (so m is even). Conjecture (originally, without an outer-boundary precondition): the projection support $\pi_D(\mathcal{C}(T))$ on the $|\gamma| = m$ inner-side spokes always contains the combined orbit*

$$\mathrm{Orbit}\big(\,(a, b, c, b, c, \ldots, b, c, a)\,\big)$$

under $S_3 \times C_m$, with a-positions at the two chord endpoints and b, c alternating elsewhere. At $m = 6$ this is the rainbow orbit of size 36 that Obs. witnessed.

The original statement is false: *the counterexample-search log (`experiments/counterexample_search.log`) records the pair*

$$T_1 = (m_1 = 3,\ (0,3),\ \mathrm{SP}), \qquad T_2 = (k_2 = 3,\ -,\ \mathrm{SP}), \qquad |\gamma| = 6,$$

where T_1 is the antipodal-chord SP tire at $\gamma = 6$ but with small outer boundary $m_1 = 3$*. Direct computation gives $|\pi_D(\mathcal{C}(T_1))| = 18$. The conjectured rainbow combined orbit has size 36, so the orbit cannot fit inside π_D; in fact only 6 of the 36 rainbow-orbit elements lie in π_D, and the literal generator pattern $(1, 2, 3, 2, 3, 1)$ is itself among the 30* missing *elements. Furthermore, $\pi_U^{(2)}(\mathcal{C}(T_2))$ has $|\pi_U| = 84$, but $\pi_D(\mathcal{C}(T_1)) \cap \pi_U(\mathcal{C}(T_2)) = \emptyset$ in both orientations, so this is also a (strict-Latin) counterexample to step-2 compatibility.*

Refined statement (now consistent with all data). *The original conjecture omitted an outer-boundary precondition. The correct conjecture is:*

> ***If additionally** $m_1 = |B^{(1)}_{\mathrm{out}}| \geq m$ **(i.e. the outer boundary is at least as long as the inner boundary), then** $\pi_D(\mathcal{C}(T))$ **contains the antipodal rainbow combined orbit.***

The threshold $m_1 \geq m$ is the same saturation threshold from step 1: when the dual cycle of T has length $n_1 = m_1 + m \geq 2m$, the spoke projection saturates the full Latin set on γ. Under this precondition all 23 pairs of the original step-2 tests and all 44 strict-Latin pairs in the search log with $m_1 \geq m$ and *$k_2 \geq m$ confirm the property.*

If the refined statement holds, this is a uniform structural property of the antipodal-chord SP tire *with sufficient outer cycle length.* The chain pigeonhole step at $|\gamma| = m$ on such a tire reduces to "π_U of the other tire intersects this fixed orbit," a much smaller compatibility claim — but only once both sides clear the outer-boundary threshold.

Direct test. At $m = 4$ ($\theta(1, 2, 2) = K_4 - e$) the antipodal-chord SP tire's π_D support has size 36 at $|\gamma| = 4$ (`tire_fiber_chords.tex`, row "(4,4) chord $(0, 2)$"), and the conjectured orbit $(a, b, c, b) \cdot S_3 \times C_4$ has size 24. Confirming the conjecture at $m = 4$ amounts to checking that this 24-element subset lies inside the 36-element support; this is mechanical.

Why antipodal? In the planar dual picture, the antipodal chord of O corresponds to the dual edge of a single "maximally-separating" chord in the tire's inner outerplanar graph: it splits π_1 of the annulus most symmetrically. Any reasonable proof of the conjecture would have to exploit this symmetry — e.g. via reflection invariance on the chord axis.

Caveats

1. Still empirical for $k \leq 6$, chord counts ≤ 2, and the 23 pairs from step 2.
2. "Same combined orbit" is taken modulo $S_3 \times C_k$ (color permutation $\times$ cyclic rotation of γ). Reflection of γ is *not* quotiented out — some orbits would coincide if it were.
3. The rainbow orbit's persistence is tied to the antipodal chord; a structural proof would need to show that the antipodal-chord SP tire's projection support π_D always contains the rainbow orbit, independently of the outer tire.