Closed-chain non-emptiness: a partial proof and the remaining gap

What this note records

In `absorption_proof.tex` we proved (via the K_3-walk parity invariant) that the forward-propagated state at L_n of a closed SR + PDS chain ending at the outer triangle is contained in the 6 permutations of $\{1,2,3\}$. This note attempts to prove the remaining "non-emptiness" half: that the state is non-empty (and hence, by S_3 invariance, equals all 6).

Result so far: the proof closes *conditionally* on a specific structural property of T_n that holds empirically but I have not proven in general.

What's easy

Lemma 1 (S_3-closure preserved by chain propagation)**.** *For any SR chain $T_1|\dots|T_n$ starting from a degenerate-inner T_1, the forward-propagated state at every L_i is closed under the S_3 action on colours.*

Proof. Each tire's joint support Π_{T_i} is determined by the proper edge 3-colourings of T_i's annular dual cycle. The S_3 action on the three colour labels acts uniformly on C_{n_i}-colourings, so Π_{T_i} is closed under the diagonal S_3 action. Forward propagation $\text{state}_{i+1} = \{\sigma_U : \exists \sigma_D \in \text{state}_i, (\sigma_U, \sigma_D) \in \Pi_{T_{i+1}}\}$ commutes with S_3: if σ is in state_{i+1} via some $\sigma_D \in \text{state}_i$, then $g \cdot \sigma$ is in state_{i+1} via $g \cdot \sigma_D \in \text{state}_i$ (using S_3-closure of state_i inductively). $\square$

Theorem 1 (Conditional non-emptiness $\Rightarrow$ exactly 6)**.** *The forward-propagated state at L_n is either empty, or equals all 6 permutations of $\{1,2,3\}$.*

Proof. By the parity invariant (proved in `absorption_proof.tex`), $\text{state}(L_n) \subseteq$ permutations of $\{1,2,3\}$. The 6 permutations form a single S_3-orbit (the S_3 action on length-3 tuples with 3 distinct colours is transitive). By Lem. 1, $\text{state}(L_n)$ is S_3-closed. An S_3-closed subset of a single S_3-orbit is either empty or the whole orbit. $\square$

So non-emptiness $\Leftrightarrow$ state $= 6$ permutations.

Non-emptiness for intermediate steps

Lemma 2 (Saturation preserves non-emptiness in the outward direction)**.** *If T_{i+1} has $m_{i+1} \geq k_{i+1}$ (the outward-PDS condition), then $\text{state}(L_{i+1})$ is non-empty whenever $\text{state}(L_i)$ is non-empty.*

Proof. By the spread-projection saturation theorem from step 1, when $m_{i+1} \geq k_{i+1}$ the σ_D-projection of $\Pi_{T_{i+1}}$ equals all of $\{1,2,3\}^{k_{i+1}}$. Hence $\text{state}(L_i) \subseteq \{1,2,3\}^{k_{i+1}} = \sigma_D$-projection of T_{i+1}, so every $\sigma_D \in \text{state}(L_i)$ is in some pair of $\Pi_{T_{i+1}}$. Thus $\text{state}(L_{i+1}) \ni \sigma_U$ for at least one σ_U paired with some σ_D in the state. Since $\text{state}(L_i) \neq \emptyset$, $\text{state}(L_{i+1}) \neq \emptyset$. $\square$

Corollary 3 (Open-chain non-emptiness)**.** *For any prefix $T_1|T_2|\dots|T_j$ of an outward-PDS chain with T_1 degenerate-inner and each subsequent T_{i+1} satisfying $m_{i+1} \geq k_{i+1}$, the state at L_j is non-empty.*

Proof. state(L_1) equals the -set from proper edge 3-colourings of C_{m_1}, which has $2^{m_1}+2(-1)^{m_1} > 0$ elements for $m_1 \geq 3$. Iterate Lem. 2. □

The remaining piece: the final step at T_n

The final step T_n has $m_n = 3$ (outer triangle). Since $|L_{n-1}| = k_n \geq 3$ in any non-trivial PDS, typically $k_n \geq 3 = m_n$ and the inequality is reversed.

When $m_n < k_n$ the saturation theorem fails: T_n's $_D$-projection is a proper subset of $\{1,2,3\}^{k_n}$. Concretely:

k	$\lvert\Pi_{T_n}\rvert$	$\lvert\sigma_D$-proj$\rvert$	$\lvert$parity set$\rvert$	$\lvert\sigma_D \cap$ parity$\rvert$
3	63	27	6	6
5	255	171	60	42
6	510	384	183	90
9	4095	3681	4920	840

Key observation. Computationally, the set of σ_D's in T_n's σ_D-projection that are *also* parity-matching equals exactly the set of σ_D's that pair with a permutation σ_U under Π_{T_n}. Call this set the "perm-paired" subset.

For non-emptiness at L_n we need: **state(L_{n-1}) intersects the perm-paired subset of T_n.**

Restatement of the remaining gap

Conjecture (Perm-paired reachability)**.** *For any SR + outward-PDS chain $T_1|\dots|T_{n-1}$ with T_1 degenerate-inner, the forward-propagated state at L_{n-1} contains at least one σ_D that is in the perm-paired subset of $T_n = (3, k_n)$ (i.e., a σ_D such that $(\sigma_U, \sigma_D) \in \Pi_{T_n}$ for some permutation σ_U).*

This is the only remaining gap. Empirically Conj. holds in every tested chain; theoretically I do not yet have a proof.

What I know about Conj.

1. By the parity invariant, state(L_{n-1}) $\subseteq$ parity-matching set on L_{n-1} (size 60 at $k = 5$).
2. The perm-paired subset has size $\leq$ |parity-matching| and is generally strictly smaller (at $k = 5$, $42 < 60$).
3. Both are S_3-closed; both are unions of S_3-orbits of size 6 (no constant orbits, since constants violate parity).
4. Empirically, at sufficiently late stages, state(L_{n-1}) = full parity-matching set. Since the perm-paired subset is strictly contained in the parity-matching set, the intersection is the perm-paired subset itself (non-empty).
5. A clean proof of Conj. would seem to require either (a) showing chain state always equals the full parity-matching set at L_{n-1}, or (b) an explicit construction of a perm-paired σ_D reachable through any outward-PDS chain.

Why it's not immediate from saturation

Lem. 2 preserves *some* non-empty state, but doesn't characterise *which* values are in the state. A more refined statement is needed: state at each L_i equals the full parity-matching set (or at least a strictly-larger-than-empty subset of perm-paired$_{T_n}$).

Why a Tait+4CT reduction is circular

In general, "state at L_n non-empty" is equivalent to the chain's underlying cubic planar graph G' admitting a proper edge 3-colouring, which by Tait's theorem is equivalent to G being 4-colourable. So Conj. for arbitrary outward-PDS chains under SR is essentially the 4CT itself (or rather, 4CT restricted to graphs admitting SR + outward-PDS decompositions).

This means Conj. cannot be proven by invoking 4CT — but a *structural* proof of it, independent of 4CT, *would constitute* a new proof of 4CT (under the SR + PDS modelling assumption).

Summary

- **Proven:** state at L_n is either empty or equals all 6 permutations of $\{1, 2, 3\}$ (Thm. 1).
- **Proven:** non-emptiness propagates through all intermediate tires under outward PDS (Cor. 3).
- **Conjectured (Conj.):** non-emptiness propagates through the final tire T_n. Empirically true; structural proof would imply 4CT.

The clean conclusion: **the chain-pigeonhole story under SR + PDS reduces to one specific reachability conjecture about chain states hitting the perm-paired subset of the final tire**. This is the sharpest version of the 4CT obstruction in our framework.