Edge 3-colorings of small outerplanar graphs with $\Delta \le 3$: a menagerie

Setup

For a graph G write $P_e(G, k)$ for the number of proper k-edge-colorings of G (= the chromatic polynomial of the line graph $L(G)$ evaluated at k). Throughout this note $\Delta(G)$ denotes the maximum degree of G; we are interested in $k = 3$ and $\Delta(G) \le 3$, with G outerplanar.

There is no *universal* closed form for $P_e(G, 3)$ on the class of subcubic outerplanar graphs, but the class is small enough that every G in it decomposes along its block–cut tree into building blocks each of which admits a closed-form count. The building blocks form a short menagerie.

The menagerie

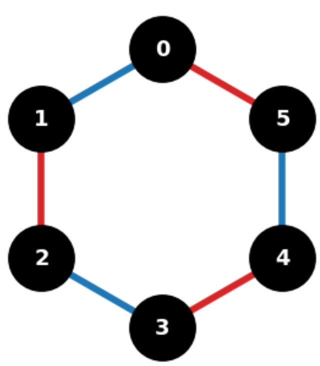
1. Path P_n (n vertices, $n-1$ edges; $\Delta \le 2$)

The line graph $L(P_n)$ is the path P_{n-1}, so

$$P_e(P_n, k) \;=\; P_{\text{vert}}(P_{n-1}, k) \;=\; k(k-1)^{n-2}, \qquad P_e(P_n, 3) \;=\; 3 \cdot 2^{n-2}.$$

(For a single edge, $n = 2$, the count is 3; for two edges in a row, $n = 3$, the count is 6.)

2. Cycle C_n ($\Delta = 2$)

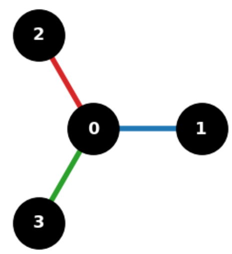


The line graph $L(C_n) = C_n$, so

$$P_e(C_n, k) \ = \ (k-1)^n + (-1)^n(k-1), \qquad P_e(C_n, 3) \ = \ 2^n + 2(-1)^n.$$

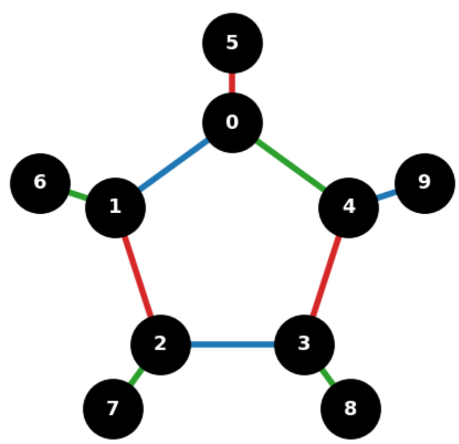
For even n the count is $2^n + 2$; for odd n it is $2^n - 2$.

3. Star $K_{1,3}$ (a single $\Delta = 3$ vertex)



Three pairwise-incident edges at a single vertex must carry three distinct colors, so $P_e(K_{1,3}, 3) = 3! = 6$. More generally $P_e(K_{1,d}, k) = k(k-1)\cdots(k-d+1)$, which is positive iff $k \geq d$, i.e. iff $k \geq \Delta(G)$.

4. Corona $C_n \circ K_1$ (cycle with one leaf per cycle vertex; $\Delta = 3$)



Each cycle vertex v has degree 3 in $C_n \circ K_1$: its two cycle edges must carry distinct colors and its leaf must carry the unique remaining third color. So the leaf coloring is *forced* by the cycle coloring, and

$$P_e(C_n \circ K_1, 3) \ = \ P_e(C_n, 3) \ = \ 2^n + 2(-1)^n.$$

This is the form of the partial tire dual $D(T)$ in the spoke-only case (with $L = n + m$).

5. Trees with $\Delta \leq 3$

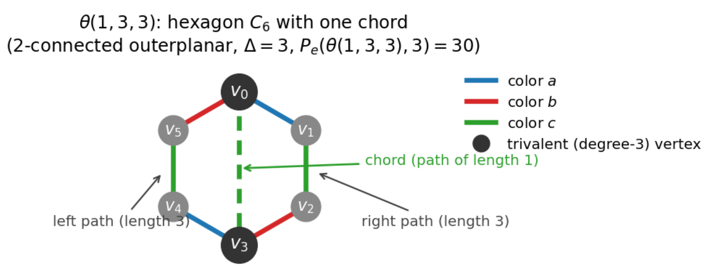
A tree T on n vertices has $|E(T)| = n - 1$ edges; its line graph $L(T)$ is a *block graph* (every block is a clique). Edge-color a tree greedily by processing edges in BFS order from a leaf: when an edge $\{u, v\}$ is added, the only colors forbidden are those already used on the edges incident to its already-colored endpoint. Hence at any vertex of degree d, when the d-th edge is added there are exactly $k - (d - 1)$ choices. For $k = 3$ and $\Delta \leq 3$:

$$P_e(T, 3) \;=\; 3 \prod_{e \in E(T) \setminus \{e_0\}} (3 - d_e),$$

where e_0 is the first edge processed and d_e is the number of already- processed edges incident to the new endpoint of e (between 1 and 2, since $\Delta \leq 3$). In practice this gives a clean product depending only on the degree sequence of T.

6. Two-connected outerplanar with $\Delta \leq 3$: cycle, possibly with a matching of chords

The 2-connected outerplanar graphs are polygons (with optional chords). Each chord raises the degree of its two endpoints by 1, and a polygon vertex already has degree 2 from the cycle. So for $\Delta \leq 3$ the chords must form a *matching* of polygon vertices — no vertex can be an endpoint of two chords. In particular the simplest non-cycle case is a polygon with a *single* chord, denoted $\theta(1, p, q)$: two paths of lengths p and q between two trivalent vertices, plus a direct edge between those two trivalent vertices. Equivalently it is C_{p+q} with a chord joining the two cycle vertices at distance p apart on the polygon.



$\theta(1, p, q)$ is outerplanar (the chord lies inside the polygon, and all polygon vertices are on the outer face) and subcubic.

Closed form. For each choice of chord color $c \in \{1, 2, 3\}$, the two cycle edges incident to each chord endpoint must lie in $\{x, y\} = \{1, 2, 3\} \setminus \{c\}$. Conditioning on whether the two pairs of cycle-edge colors at the chord endpoints agree or disagree, and using the transfer matrix $T = J - I$ on K_3 (eigenvalues 2 and -1 with multiplicities 1 and 2), one finds

$$\boxed{P_e(\theta(1, p, q),\, 3) \;=\; \frac{2^{p+q} - 2^p(-1)^q - 2^q(-1)^p + 10\,(-1)^{p+q}}{3}}.$$

Sanity checks:

- $p = q = 2$: $\theta(1, 2, 2) = K_4 - e$, formula gives $(16 - 4 - 4 + 10)/3 = 6 = P_e(K_4 - e, 3)$.
- $p = q = 3$: $\theta(1, 3, 3)$, formula gives $(64 + 8 + 8 + 10)/3 = 30$.
- $p = q = 6$: $\theta(1, 6, 6)$ (= the interior dual subgraph of the partial tire dual for the barbell-O tire of Figure 4 of the main paper), formula gives $(4096 - 64 - 64 + 10)/3 = 1326$.

The formula has been verified empirically against `Sage`'s chromatic polynomial routine for all $p, q \in \{2, 3, 4, 5, 6\}$.

General method. For $G = C_n + M$ where M is a matching of non-crossing chords, the edge chromatic polynomial can be computed by summing over chord-color assignments and applying a constrained transfer matrix to the polygon edges:

$$P_e(G,k) \;=\; \sum_{(c_1,\dots,c_r)\in[k]^r} N\big(C_n;\, \text{forbidden}(c_1,\dots,c_r),\, k\big),$$

where $N(C_n; F, k)$ counts proper k-edge-colorings of the polygon C_n subject to: for each cycle edge e_i, the colour of e_i avoids every chord-colour c_j such that an endpoint of chord j is incident to e_i (i.e. v_i or v_{i+1} is an endpoint of chord j).

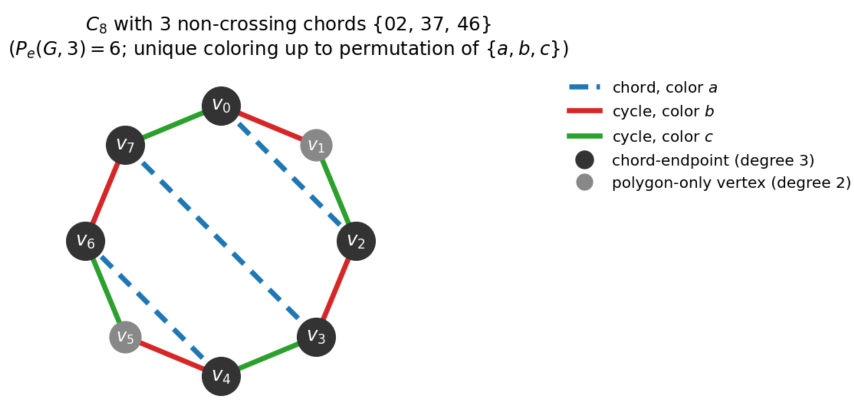
Concretely, with chord j at polygon vertices v_{a_j}, v_{b_j} and chord colour c_j, the constraints on cycle-edge colours are:

$$\begin{aligned} \text{at } v_{a_j}: &\quad c(e_{a_j-1}) \neq c_j, \quad c(e_{a_j}) \neq c_j,\\ \text{at } v_{b_j}: &\quad c(e_{b_j-1}) \neq c_j, \quad c(e_{b_j}) \neq c_j, \end{aligned}$$

on top of the usual cycle-adjacency $c(e_i) \neq c(e_{i+1})$. For each chord-colour assignment, $N(C_n; F, k)$ is a transfer-matrix product on the polygon: edge e_i's allowed colour set is $[k]\backslash\{c_j : v_i \text{ or } v_{i+1} \text{ on chord } j\}$, and the proper-coloring transitions $c(e_i) \to c(e_{i+1})$ go through a constrained $J - I$ matrix. The full sum is computed in time polynomial in n and r.

For $r = 1$ this reduces to the closed form for $\theta(1, p, q)$ above (sum over the single c_1, then transfer-matrix on the two arcs). For larger r there is no single closed form independent of the chord placement, but the computation is mechanical. The 3-chord example below illustrates how chord-colour assignments can be eliminated by direct constraint propagation when the cycle is short enough.

Example calculation (three chords on C_8). Take the polygon C_8 with the three non-crossing chords $\{(v_0, v_2),\ (v_3, v_7),\ (v_4, v_6)\}$. Vertices $v_0, v_2, v_3, v_4, v_6, v_7$ have degree 3 (chord endpoints) and v_1, v_5 have degree 2.



We compute $P_e(G, 3)$ by propagating constraints. Let c_0, c_3, c_4 denote the colors of the chords $(v_0v_2), (v_3v_7), (v_4v_6)$, and let $e_i = (v_i, v_{i+1})$ for $i = 0, \dots, 7$ (indices mod 8) be the cycle edges.

Step 1 (fix c_0). Pick the chord $0{-}2$'s color: 3 choices, say $c_0 = a$.

Step 2 (cycle edges at v_0, v_2). At v_0 the three edges $\{e_0, e_7, \text{chord } 02\}$ must use three distinct colors, so $\{c(e_0), c(e_7)\} = \{b, c\}$ where $\{a, b, c\} = \{1, 2, 3\}$. Likewise $\{c(e_1), c(e_2)\} = \{b, c\}$. The cycle constraint at v_1 ($c(e_0) \neq c(e_1)$) then forces $c(e_1) = c(e_7)$ and so $c(e_2) = c(e_0)$. Two choices for the assignment, say $c(e_0) = b$, giving $e_0 = b, e_1 = c, e_2 = b, e_7 = c$.

Step 3 (propagate to chord 37*).* At v_3: $\{c(e_2), c(e_3), c_3\}$ distinct. We have $c(e_2) = b$, and at v_7 similarly $\{c(e_6), c(e_7), c_3\}$ distinct with $c(e_7) = c$. Hence $c_3 \neq b$ and $c_3 \neq c$, forcing $c_3 = a$. Then $c(e_3) \neq b, a \Rightarrow c(e_3) = c$, and $c(e_6) \neq c, a \Rightarrow c(e_6) = b$.

Step 4 (propagate to chord 46*).* At v_4: $\{c(e_3), c(e_4), c_4\}$ distinct with $c(e_3) = c$; at v_6: $\{c(e_5), c(e_6), c_4\}$ distinct with $c(e_6) = b$. Hence $c_4 \neq c, b \Rightarrow c_4 = a$. Then $c(e_4) = b$ and $c(e_5) = c$ (the unique remaining colors).

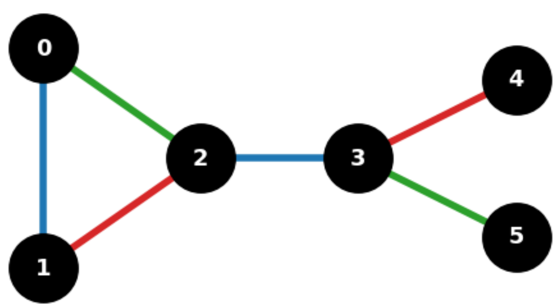
Step 5 (verify). Cycle edges go b, c, b, c, b, c, b, c around the 8-cycle (alternating, OK since 8 is even); all three chords have color a; all degree-3 vertices see three distinct colors.

Total count:

$$\#(\text{colorings}) \;=\; \underbrace{3}_{c_0 \in \{1,2,3\}} \;\cdot\; \underbrace{2}_{\text{which of } \{b,c\} \text{ is } c(e_0)} \;=\; 6.$$

This matches `Sage`'s direct chromatic-polynomial computation: $P_e(G, 3) = 6$. Up to permutation of the three colors there is a *unique* proper 3-edge-coloring of this G.

7. Block–cut decomposition



A general subcubic outerplanar graph is a union of cycle-blocks and edge-blocks glued at cut vertices. By case 6 each 2-connected block is a cycle; the remaining blocks are single edges (i.e. tree edges). At a cut vertex v of degree $d_v \in \{2, 3\}$, the colors of the d_v edges incident to v must be pairwise distinct. Counting $P_e(G, 3)$ for the whole graph G amounts to counting colorings of each block independently and then enforcing the distinct-color constraint at every cut vertex. For $k = 3$ and $\Delta \leq 3$ this gives

$$P_e(G, 3) \;=\; \prod_{B \text{ cycle block of } G} P_e(B, 3) \;\cdot\; \prod_{B \text{ edge block of } G} P_e(B, 3) \;/\; \prod_{v \text{ cut vertex}} (\text{normalization at } v),$$

where the normalization corrects the over- or under-counting at the cut-vertex constraint. For each cycle-block $B = C_n$ contributing $2^n + 2(-1)^n$ proper 3-edge-colorings, and each edge-block contributing 3, this product is computable in time linear in $|V(G)| + |E(G)|$.