PLANE DIAMOND COLORING

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Abstract.

Notation

For a coloring $C : V(G) \to S$ and a color $c \in S$, we write $C^{-1}(c) = \{v \in V(G) : C(v) = c\}$ for the preimage of c under C, i.e., the color class of c.

1. Motivation

Let G be a maximal planar graph. By the Four Color Theorem [1, 2], G admits a proper 4-coloring; the question that motivates this paper is whether such a coloring can always be exhibited via a particular structural construction, rather than as the output of an ad hoc case analysis.

The construction we have in mind proceeds as follows. Fix a root vertex $u \in V(G)$ and consider the BFS layering $\{L_0, L_1, L_2, \dots\}$ of G from u (Definition 2.1). Remove from G every edge whose two endpoints lie in the same layer L_i, producing a spanning subgraph $G^{\diamond} \subseteq G$ which we call the *diamond scaffold* of G relative to u (Definition 2.2). For any edge $\{x, y\} \in E(G)$ the BFS depths of x and y differ by at most 1, so every edge surviving in $G^{\diamond}$ joins some level L_i to L_{i+1}; in particular $G^{\diamond}$ is bipartite, and the parity of the layer index supplies a canonical proper 2-coloring of $G^{\diamond}$ using two colors, which we denote c_a and c_b.

To extend this 2-coloring of the scaffold to a proper 4-coloring of the original graph G, we must dispose of the edges discarded in passing from G to $G^{\diamond}$ — namely the edges whose endpoints share a BFS layer. The natural strategy is to recolor a chosen subset of vertices using two new colors c_c, c_d, just enough to resolve the discarded same-layer conflicts while preserving the canonical $\{c_a, c_b\}$-coloring on the remaining vertices. A *plane diamond coloring* of G (Definition 2.3) is precisely a proper 4-coloring of G obtained in this way: two of its color classes, $C^{-1}(c_a)$ and $C^{-1}(c_b)$, are confined to even-indexed and odd-indexed BFS layers respectively, so that they extend the canonical 2-coloring of the diamond scaffold $G^{\diamond}$ relative to some root u.

This paper investigates which maximal planar graphs admit such a coloring. The Four Color Theorem guarantees a proper 4-coloring; the diamond coloring asks for one obeying the additional structural constraint above. Theorem 3.3 shows the constraint genuinely fails on some triangulations, with a unique smallest obstruction of order 13; Conjecture 3.4 asserts that the obstructions disappear once $\delta(G) \geq 5$.

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2. Definitions

Definition 2.1. Let G be a graph and let $u \in V(G)$. The *distance partition* of G from u is the partition $\{L_0, L_1, L_2, \dots\}$ of $V(G)$ obtained by breadth-first search from u:

$$L_0 = \{u\}, \qquad L_{i+1} = \{v \in V(G) \setminus (L_0 \cup \cdots \cup L_i) : v \text{ is adjacent to some } w \in L_i\}.$$

Equivalently, $L_i = \{v \in V(G) : d(v, u) = i\}$, where $d(v, u)$ denotes the graph distance between v and u in G. We call each L_i the *i-th level* of the partition.

Definition 2.2. Let G be a maximal planar graph and let $u \in V(G)$, with distance partition $\{L_0, L_1, L_2, \dots\}$ from u. The *diamond scaffold* of G relative to u is the spanning subgraph $G^\diamond \subseteq G$ obtained from G by removing every edge $\{x, y\} \in E(G)$ such that $x, y \in L_i$ for some i.

Definition 2.3. Let G be a maximal planar graph. A *plane diamond coloring* of G is a proper 4-coloring C of G for which there exist a vertex $u \in V(G)$ and two distinct colors c_a, c_b such that, with respect to the distance partition $\{L_0, L_1, L_2, \dots\}$ of G from u,

$$C^{-1}(c_a) \subseteq \bigcup_{i \text{ even}} L_i \qquad \text{and} \qquad C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i.$$

3. Results

Remark 3.1. Definition 2.3 imposes a structural condition on 4-colorings of maximal planar graphs strictly stronger than the conclusion of the Four Color Theorem [1, 2]: it requires not merely the existence of a proper 4-coloring, but the existence of a proper 4-coloring together with a root u such that two of the four color classes are separated by the parity of the BFS layering from u.

Conjecture 3.2. *Every maximal planar graph G has a plane diamond coloring.*

Theorem 3.3. *The preceding conjecture is false. Moreover, the smallest counterexample has order* 13*, and is unique up to isomorphism among triangulations of order at most* 13*.*

Proof. Let G be the maximal planar graph on 13 vertices with graph6 string[1]

`LQhO?SFIwnTYTi`

shown in Figure 1. Equivalently, G has edge set

$$\{0,2\}, \{0,4\}, \{0,11\}, \{1,3\}, \{1,5\}, \{1,12\}, \{2,4\}, \{2,9\}, \{2,11\},$$
$$\{3,5\}, \{3,10\}, \{3,12\}, \{4,7\}, \{4,9\}, \{4,11\}, \{5,8\}, \{5,10\}, \{5,12\},$$
$$\{6,7\}, \{6,8\}, \{6,9\}, \{6,10\}, \{6,11\}, \{6,12\}, \{7,8\}, \{7,9\}, \{7,10\},$$
$$\{7,11\}, \{8,9\}, \{8,10\}, \{8,12\}, \{9,11\}, \{10,12\}.$$

We have $|V(G)| = 13$ and $|E(G)| = 33 = 3 \cdot 13 - 6$, so G is a triangulation.

By Definition 2.3, it suffices to show that for every root $u \in V(G)$, no proper 4-coloring C of G admits two distinct colors c_a, c_b with $C^{-1}(c_a)$ contained in the union of even-indexed levels and $C^{-1}(c_b)$ contained in the union of odd-indexed levels of the distance partition from u.

[1] We use the standard graph6 encoding of McKay; see [3].

For a fixed root u, the existence of such a triple (C, c_a, c_b) is equivalent to 4-colorability of the auxiliary graph H_u obtained from G by adjoining two new vertices α, β, joining α to every vertex in odd-indexed levels, joining β to every vertex in even-indexed levels, and adding the edge $\{\alpha, \beta\}$. Indeed, in any proper 4-coloring of H_u the colors of α and β are distinct and absent from the odd-parity and even-parity layers of G respectively, yielding $c_a := C(\alpha)$ and $c_b := C(\beta)$. Conversely, given a 4-coloring satisfying the parity-separation condition, setting $C(\alpha) := c_a$ and $C(\beta) := c_b$ extends it to a proper 4-coloring of H_u.

A direct computation (using `Sage`'s `chromatic_number`) verifies that $\chi(H_u) > 4$ for every $u \in V(G)$, so G admits no plane diamond coloring.

For minimality and uniqueness, we exhaustively enumerated every maximal planar graph of order at most 13 using `Sage`'s `graphs.planar_graphs` generator (with `minimum_connectivity=3` and `maximum_face_size=3`). The numbers of triangulations $1, 1, 2, 5, 14, 50, 233, 1249, 7595, 49566$ at orders $4, 5, \ldots, 13$ respectively (matching OEIS A000109) were each tested for the existence of a plane diamond coloring, and exactly one — the graph G above, occurring at order 13 — was found to lack one. $\square$

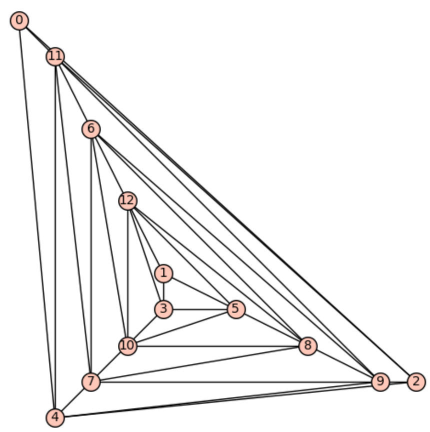

FIGURE 1. The unique smallest maximal planar graph with no plane diamond coloring; it has 13 vertices and degree sequence $(6, 6, 6, 6, 6, 6, 6, 5, 5, 4, 4, 3, 3)$.

Conjecture 3.4. *Every maximal planar graph G of minimum degree at least 5 has a plane diamond coloring.*

Theorem 3.5. *Conjecture 3.4 is false. The smallest counterexamples have order 28, and every maximal planar graph of minimum degree at least 5 and order at most 27 admits a plane diamond coloring.*

Proof. By exhaustive enumeration via `Sage`'s `graphs.planar_graphs` generator (with `minimum_connectivity=3`, `maximum_face_size=3`, and `minimum_degree=5`) and the auxiliary-graph reduction described in the proof of Theorem 3.3, every maximal planar graph of minimum degree at least 5 and order in $\{12, 13, \ldots, 27\}$

admits a plane diamond coloring. The numbers of such triangulations at orders $12, 13, \ldots, 27$ are

$$1,\ 0,\ 1,\ 1,\ 3,\ 4,\ 12,\ 23,\ 73,\ 192,\ 651,\ 2070,\ 7290,\ 25381,\ 91441,\ 329824,$$

totalling 456,967 graphs, none of which is a counterexample.

At order 28, however, counterexamples do exist. The graph in Figure 2 is one such, with canonical graph6 string

[??DAaGP@OA_AI@DCPOaI_gh@PO?????C??B???|C?CIG?GIA?iD@?TPC?VQG_Bi.

It has $|V| = 28$, $|E| = 78 = 3 \cdot 28 - 6$, minimum degree 5, and chromatic number 4. Two further counterexamples at order 28 have canonical graph6 strings

[?‘???I@PCAG????@COGaGA_OD?DD?Aa_AII?PPV???Y??@ii?ATT?@T?T@agAgX

and

[??DAaGP@OA_AI@DCPOaI_gh@PO?????C??BIA??gG?PC?IPC?Ig_?tIG?TO??F~.

Direct computation (using Sage's chromatic_number) verifies $\chi(H_u) > 4$ for every u in each of these graphs. $\square$

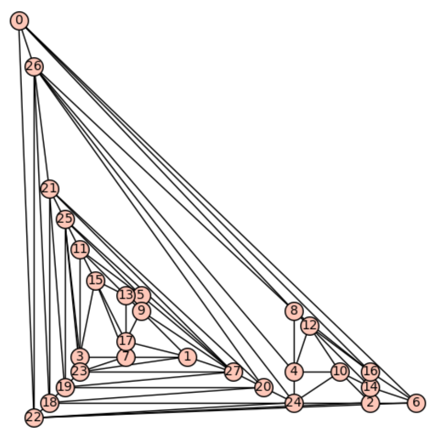

FIGURE 2. One of three known smallest counterexamples to Conjecture 3.4: a maximal planar graph on 28 vertices with minimum degree 5 admitting no plane diamond coloring.

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[3] B. D. McKay, *Description of graph6, sparse6 and digraph6 encodings*, https://users.cecs.anu.edu.au/~bdm/data/formats.txt.