PLANE DEPTH SEQUENCING

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ABSTRACT.

1. DEFINITIONS

Definition 1.1. Let G be a graph with a plane embedding, and let C be the outer cycle of that embedding. The *plane depth* of a vertex $v \in V(G)$ relative to the embedding and C is

$$\operatorname{depth}(v) = \min_{u \in V(C)} d(v,u),$$

where $d(v,u)$ denotes the graph distance between v and u in G.

Definition 1.2. An edge $\{u,v\} \in E(G)$ is a *level edge* if $\operatorname{depth}(u) = \operatorname{depth}(v)$.

Definition 1.3. A triangle $\{u,v,w\}$ in G is an *up triangle* if the multiset of depths of its vertices is $\{d, d+1, d+1\}$ for some $d \geq 0$, a *down triangle* if the multiset of depths is $\{d, d, d+1\}$ for some $d \geq 0$, and a *neutral triangle* if the multiset of depths is $\{d,d,d\}$ for some $d \geq 0$.

Remark 1.4. We now relate our terminology to existing terminology, namely k-outerplanar graphs [1]. The following definition and lemma show that the subgraph induced by any single depth level is outerplanar, i.e., 1-outerplanar in the sense of Baker.

Definition 1.5. A plane graph is *outerplanar* if every vertex lies on the outer face. More generally, a plane graph is *k-outerplanar* for $k \geq 1$ if removing all vertices on the outer face yields a $(k-1)$-outerplanar graph, where every graph on the empty vertex set is 0-outerplanar.

Lemma 1.6. *Let G be a graph with a plane embedding and outer cycle C. For each $d \geq 0$, the subgraph of G induced by $V_d = \{v \in V(G) : \operatorname{depth}(v) = d\}$ is outerplanar.*

Proof. Let $H = G[V_d]$ with the plane embedding inherited from G. It suffices to show every vertex of H lies on the outer face of H.

For $d = 0$, we have $V_0 = C$, so H is outerplanar.

For $d \geq 1$, let U be the open subset of the plane obtained by removing all vertices and edges of H. We show every $v \in V_d$ lies on the boundary of the component U_{out} of U containing the outer face of G.

Since every vertex in $V_{\leq d-1}$ has a shortest path to C passing entirely through $V_{\leq d-1}$, the subgraph $G[V_{\leq d-1}]$ is connected and contains C. Its vertices and edges lie in U (as they are not in H), and C borders the outer face of G, so $G[V_{\leq d-1}]$ and the outer face of G are connected within U, hence both lie in U_{out}.

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Now let $v \in V_d$. Since $\operatorname{depth}(v) = d \geq 1$, there exists $u \in V_{d-1}$ adjacent to v in G. The edge $\{v, u\}$ is not an edge of H, so it lies in U. Since $u \in V_{d-1} \subset U_{\text{out}}$ and $\{v, u\}$ is a connected subset of U containing u, the entire edge lies in U_{out}. The vertex v is an endpoint of this edge but is not in U, so v lies on the boundary of U_{out}, i.e., on the outer face of H. □

Definition 1.7. Let G be a maximal planar graph with a plane embedding and outer cycle C. The *deep embedding* of G is the graph G' obtained from G by the following operation: for every 3-cycle $\{u, v, w\} \subseteq V(G)$ such that

$$\operatorname{depth}(u) = \operatorname{depth}(v) = \operatorname{depth}(w),$$

add a new vertex x to G adjacent to each of u, v, and w.

Lemma 1.8. *Let G' be the deep embedding of a maximal planar graph G. Every face of G' is either an up triangle or a down triangle.*

Proof. We first establish that for any edge $\{p, q\}$ in G, the depths of p and q differ by at most 1. Suppose for contradiction that $\operatorname{depth}(p) = d$ and $\operatorname{depth}(q) = d + n$ for some $n \geq 2$. Since $\operatorname{depth}(p) = d$, there exists a path of length d from p to some vertex of C. Prepending the edge $\{q, p\}$ gives a path of length $d + 1$ from q to C, so $\operatorname{depth}(q) \leq d + 1 < d + n$, a contradiction. The case $\operatorname{depth}(q) = d - n$ is handled identically: there exists a path of length $d - n$ from q to some vertex of C, and prepending the edge $\{p, q\}$ gives a path of length $d - n + 1 \leq d - 1 < d$ from p to C, contradicting $\operatorname{depth}(p) = d$.

Since G is a triangulation, every interior face of G is a triangle $\{u, v, w\}$ with all three pairs adjacent. By the above, each pair of vertices in a triangle differs in depth by at most 1, so no triangle can contain vertices of depths d and $d + 2$ simultaneously. The possible depth patterns for a triangle in G are therefore exactly a neutral triangle, a down triangle, or an up triangle.

We now consider each case under the deep embedding.

Case 1: up triangle or down triangle. These triangles are not modified by the deep embedding, so they remain as faces of G', satisfying the lemma.

Case 2: neutral triangle. The deep embedding inserts a new vertex x adjacent to u, v, and w, replacing the face $\{u, v, w\}$ with three new faces $\{u, v, x\}$, $\{v, w, x\}$, and $\{u, w, x\}$. It remains to determine the depth of x in G'. Since x is adjacent only to u, v, and w, every path in G' from x to C must pass through one of them, so x has strictly greater depth than u, v, and w. Each of the three new faces is thus a down triangle, satisfying the lemma.

Since every face of G' falls into one of these cases, the result follows. □

References

[1] B. S. Baker, *Approximation algorithms for NP-complete problems on planar graphs*, Journal of the ACM, vol. 41, no. 1, pp. 153–180, 1994.