DUAL DECOMPOSITION OF MINIMAL COUNTEREXAMPLES

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Abstract.

1. The minimal counterexample

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

Lemma 1.1 (Reduction to triangulations)**.** *If every triangulation is properly* 4*-vertex-colourable, then so is every plane graph.*

Proof. Let H be a plane graph. Add edges to H, maintaining planarity, until no further edge can be added; the result is a triangulation H^+ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of H^+ restricts to a proper 4-colouring of H, since every edge of H is an edge of H^+. □

By Lemma 1.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

Definition 1.2 (Minimal counterexample)**.** Let G be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call G a *minimal counterexample.* By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

Remark 1.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being K_4), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

Lemma 1.4 (Minimum degree)**.** *A minimal counterexample G has minimum degree $\delta(G) \geq 5$.*

Proof. Suppose some vertex v has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then G' is a plane graph on fewer vertices, so by Definition 1.2 and Lemma 1.1 it has a proper 4-colouring. The at most three neighbours of v use at most three colours, so a fourth colour is free for v, extending the colouring to G — a contradiction.

If $d = 4$, again 4-colour $G - v$. If the four neighbours of v use at most three colours we extend as before, so assume they receive all four colours; let v_1, v_2, v_3, v_4 be the neighbours in cyclic order around v, coloured $1, 2, 3, 4$. Consider the subgraph

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induced by the colour classes 1 and 3, and let K be its connected component containing v_1. If $v_3 \notin K$, swap colours 1 and 3 on K; now no neighbour of v is coloured 1, freeing it for v. If $v_3 \in K$, then a 1–3 Kempe chain joins v_1 to v_3, and this chain together with v encloses exactly one of v_2, v_4; hence the 2–4 component containing v_2 cannot also reach v_4, and swapping colours 2 and 4 on it frees colour 2 for v. Either way the colouring extends to G, a contradiction.

Hence $\delta(G) \geq 5$. $\square$

2. The reduced dual

Write G' for the dual of G: since G is a triangulation, G' is a cubic plane graph in which each vertex of G corresponds to a face of G', each face of G to a vertex of G', and each edge to a dual edge. A vertex of G of degree k corresponds to a k-gonal face of G'.

By Lemma 1.4, $\delta(G) \geq 5$, and Euler's formula gives $\sum_{u \in V(G)} (6 - \deg u) = 12$, so G has a vertex of degree exactly 5 (indeed at least twelve). Fix such a vertex v. Its dual face F_v is a pentagon, bounded by the five dual vertices corresponding to the five faces of G incident to v.

Definition 2.1 (Reduced dual)**.** Let v be a degree-5 vertex of G with pentagonal dual face F_v, and fix an index $i \in \{0, 1, 2, 3, 4\}$. The *reduced dual* $\widehat{G}'_{v,i}$ is the plane graph obtained from G' as follows.

1. Delete the five dual vertices on the boundary of F_v, together with all edges incident to them. Each deleted vertex is cubic, with two edges on ∂F_v and one edge leaving F_v; deleting the five boundary vertices therefore removes the five external edges as well, dropping their five outer endpoints from degree 3 to degree 2. These five degree-2 vertices lie on the boundary of a single face F of the resulting graph.
2. List the five degree-2 vertices in clockwise order around F as $A = (A_0, A_1, A_2, A_3, A_4)$.
3. Add a new vertex v_n and join it to A_i, A_{i+1}, and A_{i+2} (indices mod 5) by three new edges.
4. Add a new edge between A_{i+3} and A_{i+4} (indices mod 5).

Remark 2.2. Steps (3) and (4) restore cubicity: A_i, A_{i+1}, A_{i+2} each gain one edge to v_n and A_{i+3}, A_{i+4} each gain the new edge, so all five return to degree 3, and v_n has degree 3. Since $A_i, \ldots, A_{i+2}$ and A_{i+3}, A_{i+4} are each consecutive along ∂F, the new vertex and edge can be drawn inside F without crossings, so $\widehat{G}'_{v,i}$ is again a cubic plane graph. The construction depends on the choice of i up to the rotational symmetry of A.

Definition 2.3 (Edges of the reduced dual)**.** The four edges added in steps (3) and (4) of Definition 2.1 are named as follows. The chord $A_{i+3}A_{i+4}$ is the *merged edge*; the edge $A_{i+1}v_n$ is the *spike edge*; the edge A_iv_n is the *side-*0 *edge*; and the edge $A_{i+2}v_n$ is the *side-*1 *edge*. In the $i = 0$ case of Figure 1 these are $\{A_3, A_4\}$, $\{A_1, v_n\}$, $\{A_0, v_n\}$, and $\{A_2, v_n\}$ respectively.

We will use the following structural fact about proper 3-edge-colourings near a pentagonal face of a cubic plane graph; it is stated for a generic such graph H, not specifically for the reduced dual.

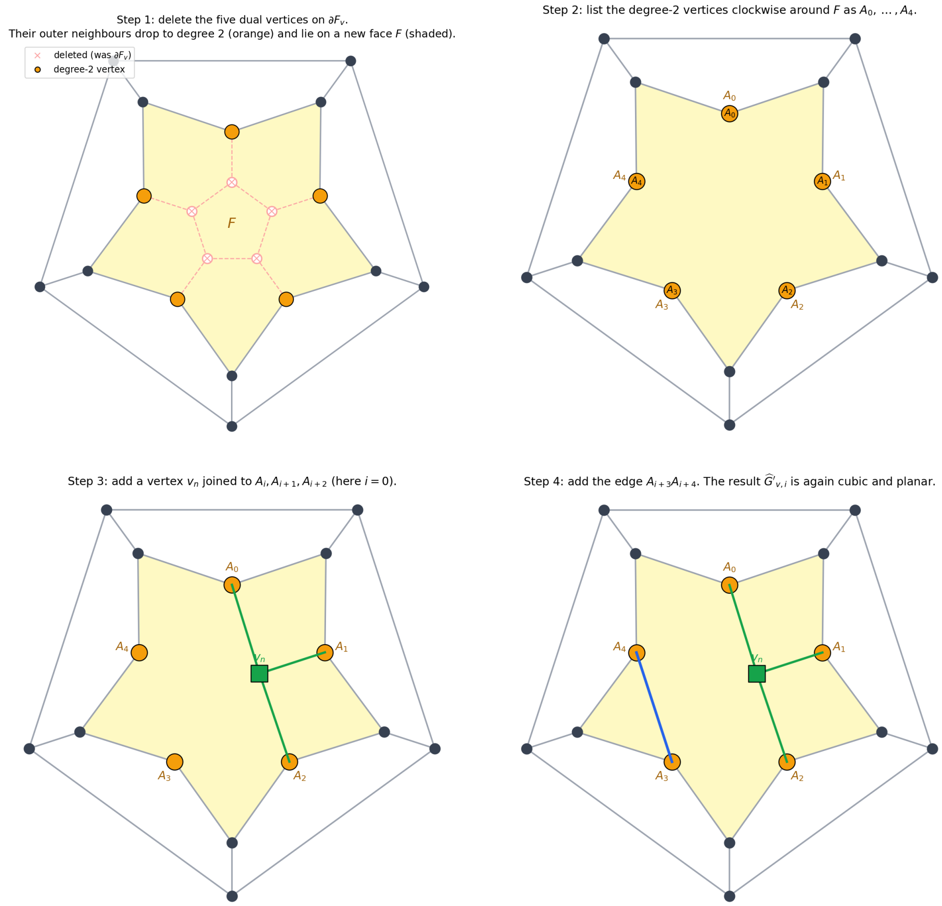

FIGURE 1. The four steps of Definition 2.1, illustrated on G' = the dodecahedron (dual of the icosahedron) with F_v the inner pentagon and $i = 0$. Top left: delete the five boundary vertices of F_v, leaving five degree-2 vertices on a new face F. Top right: order them clockwise as $A_0, \ldots, A_4$. Bottom left: add v_n joined to A_0, A_1, A_2. Bottom right: add the chord A_3A_4, giving the cubic plane graph $\widehat{G}'_{v,0}$.

Lemma 2.4 (Pentagonal externals)**.** *Let H be a cubic plane graph and F a pentagonal face of H, with ∂F traversed clockwise as u_0, u_1, u_2, u_3, u_4. For each i let f_i be the unique edge of H incident to u_i that does not lie on ∂F. An assignment φ of colours from $\{1, 2, 3\}$ to the ten edges incident to $\{u_0, \ldots, u_4\}$ is proper at every u_i if and only if there is some index j such that*

$$\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2}) \quad \text{and} \quad \{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1, 2, 3\} \setminus \{\varphi(f_j)\},$$

indices mod 5*.*

Proof. Write $e_i = u_iu_{i+1}$ for the boundary edges of ∂F (indices mod 5). A colouring φ is proper at every u_i if and only if at each u_i the three incident edges e_{i-1}, e_i, f_i

receive three distinct colours; whenever this holds, $\varphi(f_i)$ is forced to be the unique colour in $\{1,2,3\} \setminus \{\varphi(e_{i-1}), \varphi(e_i)\}$, and φ restricts to a proper 3-edge-colouring of the cycle ∂F.

(⇒) The line graph of ∂F is C_5, whose maximum independent set has size 2, so no colour appears more than twice on ∂F; and since ∂F is an odd cycle, all three colours appear. The colour multiset on $(\varphi(e_0), \ldots, \varphi(e_4))$ is therefore $(2,2,1)$, with the singleton at a unique position. Cyclically shifting indices we may place this position at 0; let c be the singleton colour. The remaining four edges form the path $e_1e_2e_3e_4$, which by propriety alternates between the other two colours, so for some labelling $\{a,b,c\} = \{1,2,3\}$,

$$(\varphi(e_0), \varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)) = (c,a,b,a,b).$$

Reading off the forced values of $\varphi(f_i)$,

$$\varphi(f_0) = a, \quad \varphi(f_1) = b, \quad \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c,$$

which is the lemma's pattern at $j = 2$ (the cyclic shift maps this back to the corresponding j in the original indexing). This case is the unique proper 3-edge-colouring of ∂F up to cyclic shift and permutation of $\{1,2,3\}$ (since $5 \cdot 3! = 30 = P(C_5, 3)$, the chromatic polynomial of C_5 at 3), so it exhausts every proper φ.

(⇐) The lemma's hypothesis is invariant under cyclic shifts of indices and under permutations of $\{1,2,3\}$, so we may assume $j = 2$, $\varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c$, $\varphi(f_0) = a$, and $\varphi(f_1) = b$, with $\{a,b,c\} = \{1,2,3\}$. Propriety at u_i and u_{i+1} requires $\varphi(e_i) \notin \{\varphi(f_i), \varphi(f_{i+1})\}$, which gives

$$\varphi(e_0) = c, \quad \varphi(e_1) = a, \quad \varphi(e_2) \in \{a,b\}, \quad \varphi(e_3) \in \{a,b\}, \quad \varphi(e_4) = b.$$

The remaining propriety condition $\varphi(e_{i-1}) \neq \varphi(e_i)$ holds automatically at u_0, u_1, u_4, forces $\varphi(e_2) = b$ at u_2, and then forces $\varphi(e_3) = a$ at u_3. The resulting triples $(\varphi(e_{i-1}), \varphi(e_i), \varphi(f_i))$ at u_0, u_1, u_2, u_3, u_4 are

$$(b,c,a), \quad (c,a,b), \quad (a,b,c), \quad (b,a,c), \quad (a,b,c),$$

each a permutation of $\{1,2,3\}$, so φ is proper at every u_i. □

Remark 2.5. The two-element condition $\{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1,2,3\} \setminus \{\varphi(f_j)\}$ cannot be dropped: a 3-colouring satisfying $\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2})$ alone need not extend, e.g. $(1,1,1,1,2)$.

Since $\widehat{G}'_{v,i}$ is the dual of a triangulation on fewer vertices than G, it is 3-edge-colourable by the minimality of G. The following lemma constrains every such colouring.

Lemma 2.6. *Let G be a minimal counterexample, and let $\widehat{G}'_{v,i}$ be a reduced dual of its dual G'. Then in every proper 3-edge-colouring of $\widehat{G}'_{v,i}$, the merged edge and the spike edge receive the same colour.*

Proof. After cyclically relabelling, assume $i = 0$. Suppose for contradiction that φ is a proper 3-edge-colouring of $\widehat{G}'_{v,0}$ in which the merged edge $\{A_3, A_4\}$ and the spike edge $\{A_1, v_n\}$ receive different colours (Figure 2, top), and write

$$X = \varphi(\{A_0, v_n\}), \quad Y = \varphi(\{A_1, v_n\}), \quad Z = \varphi(\{A_2, v_n\}), \quad W = \varphi(\{A_3, A_4\}).$$

Propriety of φ at v_n forces $\{X,Y,Z\} = \{1,2,3\}$, and the assumption $W \neq Y$ leaves $W \in \{X, Z\}$.

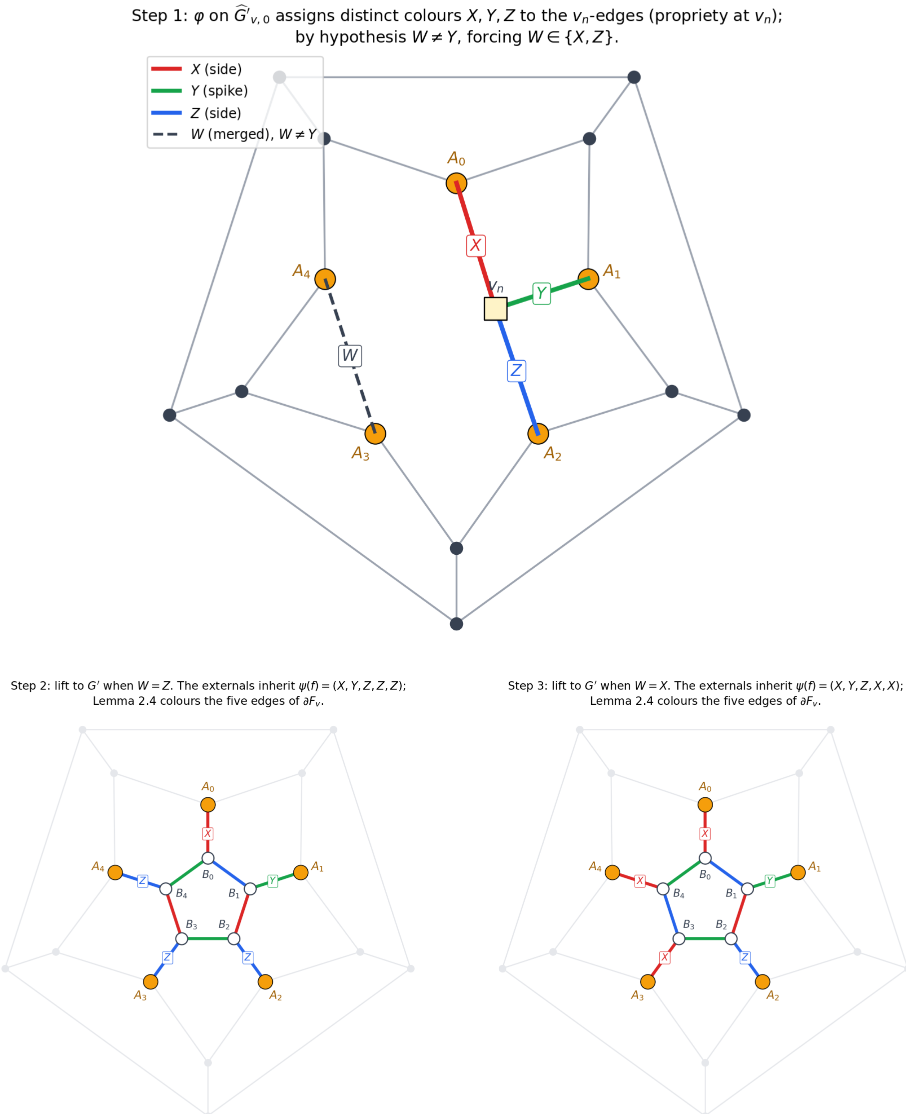

FIGURE 2. The proof of Lemma 2.6, illustrated for $i = 0$ on G' = the dodecahedron. Top: under the assumption $W \neq Y$, propriety at v_n forces $W \in \{X, Z\}$. Bottom: in either case the lift to G' has externals satisfying the hypothesis of Lemma 2.4, which colours ∂F_v to extend ψ to a proper 3-edge-colouring of G'.

We lift φ to a colouring ψ of $E(G')$ as follows. Let $B_0, \ldots, B_4$ be the boundary vertices of ∂F_v in G', indexed so that $f_k = B_k A_k$. On every edge of G' that survived the reduction, set $\psi = \varphi$. At each A_k the two surviving edges retain their φ-colours, so the remaining edge at A_k — in G' this is the external f_k; in $\widehat{G}'_{v,0}$ this is a v_n-edge ($k \in \{0, 1, 2\}$) or the chord ($k \in \{3, 4\}$) — is forced to take the third colour at A_k. Since the two-surviving-edge colours at A_k agree in G' and $\widehat{G}'_{v,0}$, the third colour does too, giving

$$\psi(f_0) = X, \quad \psi(f_1) = Y, \quad \psi(f_2) = Z, \quad \psi(f_3) = \psi(f_4) = W$$

(the last two equalities holding because the chord is a single edge contributing its colour at each of A_3 and A_4).

It remains to assign colours to the five boundary edges B_kB_{k+1} of ∂F_v. Apply Lemma 2.4 to G' at the face F_v with the B_k's as its boundary vertices and the same indexing. The external vector $(\psi(f_0), \ldots, \psi(f_4)) = (X, Y, Z, W, W)$ falls into one of two cases (Figure 2, bottom):

- if $W = Z$, it is (X, Y, Z, Z, Z): three consecutive Z's at positions $2, 3, 4$, with $\{X, Y\} = \{1, 2, 3\} \setminus \{Z\}$;
- if $W = X$, it is (X, Y, Z, X, X): three consecutive X's at positions $3, 4, 0$, with $\{Y, Z\} = \{1, 2, 3\} \setminus \{X\}$.

Each case satisfies the hypothesis of Lemma 2.4; its ($\Leftarrow$) direction therefore assigns colours to the boundary edges B_kB_{k+1} that make ψ proper at every B_k.

The resulting ψ is a proper 3-edge-colouring of G': proper at every B_k by the lemma, at every A_k by the construction, and at every other vertex because such a vertex has the same neighbourhood in G' as in $\widehat{G}'_{v,0}$ with the same incident-edge colours. By Tait's theorem, G' is 3-edge-colourable iff G is 4-vertex-colourable, contradicting that G is a counterexample. The assumption $W \neq Y$ is therefore false. $\square$

For a pair of colours $\{a, b\} \subseteq \{1, 2, 3\}$, the subgraph of $\widehat{G}'_{v,i}$ on the edges coloured a or b is 2-regular (since at each vertex exactly one of the three incident edges is excluded), and hence a disjoint union of cycles. We call each such cycle a $\{a, b\}$-*Kempe cycle*, and reserve the notation for the specific cycle containing a given edge when the context makes it clear. Swapping the two colours on a single Kempe cycle yields another proper 3-edge-colouring of the same graph.

Lemma 2.7 (Kempe cycles through the spike)**.** *Let G be a minimal counterexample, fix a reduced dual $\widehat{G}'_{v,i}$ of G', and let φ be a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Write c for the common colour assigned by φ to the spike and the merged edge (Lemma 2.6), and c_0, c_1 for the colours of the side-0 and side-1 edges respectively, so $\{c, c_0, c_1\} = \{1, 2, 3\}$. Then*

1. *the $\{c, c_0\}$-Kempe cycle through the spike edge contains both the side-0 edge and the merged edge;*
2. *the $\{c, c_1\}$-Kempe cycle through the spike edge contains both the side-1 edge and the merged edge.*

Proof. We prove (1); (2) is the same argument with c_1 and the side-1 edge in place of c_0 and the side-0 edge.

The spike edge $\{A_{i+1}, v_n\}$ and the side-0 edge $\{A_i, v_n\}$ share the vertex v_n and receive the two colours c, c_0, so they both lie on the $\{c, c_0\}$-Kempe cycle through v_n. Suppose for contradiction that the merged edge lies on a different $\{c, c_0\}$-Kempe cycle K (it lies on *some* such cycle, since it has colour c). Let φ' be obtained from φ by swapping the colours c and c_0 along K alone: this is a Kempe swap, so φ' is again a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Under φ' the spike edge — which is not on K — still has colour c, but the merged edge — which is on K — now has colour c_0.

Hence in φ' the spike and the merged edge receive distinct colours, contradicting Lemma 2.6 applied to φ'. □

3. An iterated reduction

The reduced-dual construction in Definition 2.1 can be iterated: starting from a proper 3-edge-colouring φ_1 of a reduced dual $\widehat{G}'_{v,i}$, we apply the construction again to that graph at a pentagonal face whose ten incident edges avoid the four named edges from the first reduction, extending φ_1 across the new reduction. The protected edges accumulate into a set E that grows by four per iteration, and the process terminates when E has blocked every pentagonal face.

Algorithm 3.1 (Iterated reduction with protected edges)**.** Let G be a triangulation we assume to be a minimal counterexample to the Four Colour Theorem. The algorithm produces a sequence $H_1, H_2, \dots$ of cubic plane graphs, proper 3-edge-colourings φ_t of H_t, and a growing set E of protected edges.

(0) Form $G' := \mathrm{dual}(G)$, a cubic plane graph.
(1) Choose a degree-5 vertex v of G (equivalently a pentagonal face F_v of G') and an index $i_1 \in \{0, \dots, 4\}$. Apply Definition 2.1 to form $H_1 := \widehat{G'}_{v,i_1}$, and fix any proper 3-edge-colouring φ_1 of H_1 (one exists by the minimality of G).
(2) Initialise $E := \{\text{spike}, \text{side-0}, \text{side-1}, \text{merged}\}$, the four named edges of the reduction in (1).
(3) (Iterate.) At step $t \geq 2$, given H_{t-1}, φ_{t-1}, and $E \subseteq E(H_{t-1})$:
   (a) Find a pentagonal face F of H_{t-1} whose ten incident edges — the five boundary edges of ∂F and the five external edges at ∂F — are all outside E. If no such F exists, terminate.
   (b) By Lemma 2.4 applied to H_{t-1} at F under φ_{t-1}, the external vector has shape (a, b, c, c, c) up to cyclic rotation. Choose an index i_t for which $\varphi_{t-1}(f_{i_t+3}) = \varphi_{t-1}(f_{i_t+4})$ and $\varphi_{t-1}(f_{i_t}), \varphi_{t-1}(f_{i_t+1}), \varphi_{t-1}(f_{i_t+2})$ are three distinct colours.
   (c) Apply Definition 2.1 to H_{t-1} at (F, i_t) to form H_t.
   (d) Extend φ_{t-1} to a proper 3-edge-colouring φ_t of H_t: every surviving edge keeps its φ_{t-1}-colour, and each new edge takes the unique colour completing the palette at its endpoint (consistent across both endpoints of the chord by the choice of i_t).
   (e) Add the four named edges of the step-t reduction to E.
(4) Repeat (3) until termination.

Remark 3.2. At each iteration, $|V(H_t)| = |V(H_{t-1})| - 4$ and $|E(H_t)| = |E(H_{t-1})| - 6$, so H_t shrinks at a fixed rate; the protected set $|E|$ grows by exactly four; and every protected edge survives all subsequent reductions. Since the graph is finite, termination is guaranteed. By Lemma 2.4, step (b) never fails: some valid i_t always exists for any pentagonal face under any proper colouring. Termination is therefore combinatorial: it occurs precisely when E touches every pentagonal face of H_{t-1}.

Remark 3.3. Lemma 2.6 applies only at $t = 1$, when H_1 is a reduced dual of G'. For $t \geq 2$, H_t is a reduced dual of H_{t-1} rather than of G', and H_{t-1} is itself 3-edge-colourable, so the non-3-edge-colourability argument that drives Lemma 2.6 does not carry over. Whether the constraints accumulated in E propagate any further structure to φ_t for $t \geq 2$ is left open.

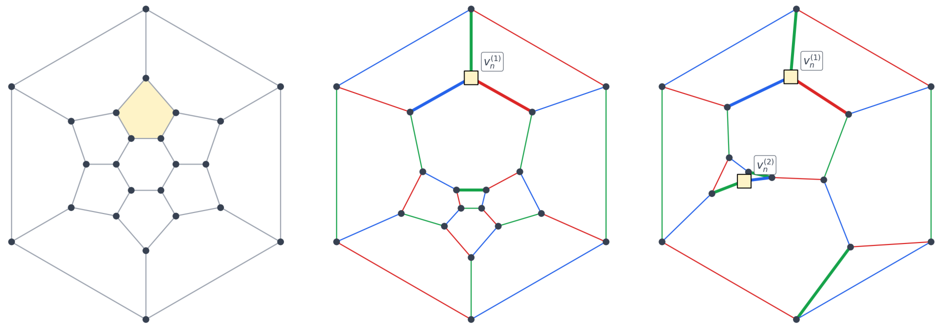

FIGURE 3. Algorithm 3.1 on $G' = \text{dual}(G)$, where G is the first min-degree-5 plantri triangulation on 14 vertices and φ_1 is a specific proper 3-edge-colouring of H_1 that satisfies both the chord-apex condition (Lemma 2.6) and the Kempe-cycle condition (Lemma 2.7), found by `experiments/search_kempe_property.py`. *Left:* G' (24 vertices, 36 edges) with the chosen pentagonal face shaded. *Centre:* H_1 (20 vertices, 30 edges) after step (1) with $i_1 = 1$, 3-edge-coloured by φ_1; the four edges around $v_n^{(1)}$ in E are drawn thicker, and the spike and merged edges share the colour green. *Right:* H_2 (16 vertices, 24 edges) after step (3) with $i_t = 3$; eight edges are protected, and the algorithm terminates one step later (no remaining safe pentagonal face in H_2). The generating script is `experiments/draw_iterated_reduction_n14.py`; layouts are Tutte barycentric embeddings with the outer face picked to keep $v_n^{(1)}, v_n^{(2)}$ in the interior.

Lemma 3.4 (Exactly one matching pair in the algorithm's output)**.** *Let G be a minimal counterexample to the Four Colour Theorem, and let (H_{t^*}, φ_{t^*}) be the final graph-and-colouring produced by some terminating execution of Algorithm 3.1 on G, with named pairs $(\text{spike}_t, \text{merged}_t)$ for $t = 1, \dots, t^*$. Then there is exactly one t with $\varphi_{t^*}(\text{spike}_t) = \varphi_{t^*}(\text{merged}_t)$, and it is $t = 1$.*

Proof. The algorithm never re-colours an existing edge: at each iteration step (3d) every surviving edge keeps its φ_{t-1}-colour, and the four new edges receive fresh colours forced by propriety. Hence for every $1 \le k \le t \le t^*$,

$$\varphi_t(\text{spike}_k) = \varphi_k(\text{spike}_k), \qquad \varphi_t(\text{merged}_k) = \varphi_k(\text{merged}_k);$$

the colours of the step-k named edges, once written, are permanent. It suffices to compare $\varphi_k(\text{spike}_k)$ and $\varphi_k(\text{merged}_k)$ at the step where each pair is introduced.

Case $k = 1$. Since G is a minimal counterexample, H_1 is a reduced dual of G'. Lemma 2.6 applied to φ_1 gives $\varphi_1(\text{spike}_1) = \varphi_1(\text{merged}_1)$.

Case $k \ge 2$. At step k the algorithm picks an index i_k for which $f_{i_k+3} = f_{i_k+4}$ (chord consistency) and $f_{i_k}, f_{i_k+1}, f_{i_k+2}$ are pairwise distinct (propriety at the new v_n), where f is the external vector of the chosen pentagonal face of H_{k-1} under φ_{k-1}. Step (3d) then assigns

$$\varphi_k(\text{spike}_k) = f_{i_k+1}, \qquad \varphi_k(\text{merged}_k) = f_{i_k+3}.$$

By Lemma 2.4, f has the $(2,2,1)$ pattern: a block of three consecutive positions $\{p, p+1, p+2\} \pmod 5$ on which it is constantly some colour c, while the remaining two positions $\{p+3, p+4\}$ hold the two non-c colours, one each. The condition $f_{i_k+3} = f_{i_k+4}$ forces (i_k+3, i_k+4) to be either $(p, p+1)$ or $(p+1, p+2)$ — the two consecutive pairs inside the block — and correspondingly $i_k + 1 \in \{p+3, p+4\}$, *outside* the block. So f_{i_k+1} is not c, whereas $f_{i_k+3} = c$, and hence $\varphi_k(\mathrm{spike}_k) \neq \varphi_k(\mathrm{merged}_k)$.

Combining the two cases, exactly one $t \in \{1, \ldots, t^*\}$ — namely $t = 1$ — has $\varphi_{t^*}(\mathrm{spike}_t) = \varphi_{t^*}(\mathrm{merged}_t)$. □

Lemma 3.5 (All-distinct colouring exists on a 4-colourable graph)**.** *Let G be a* 4*-colourable maximal planar graph of minimum degree* ≥ 5 *(equivalently, a maximal planar graph that is* not *a minimal counterexample to the Four Colour Theorem). Then there is an execution of Algorithm 3.1 on G whose final colouring φ_{t^*} satisfies $\varphi_{t^*}(\mathrm{spike}_t) \neq \varphi_{t^*}(\mathrm{merged}_t)$ for every $t \in \{1, \ldots, t^*\}$. In particular, there exists a proper* 3*-edge-colouring of H_{t^*} under which every spike-merged pair has distinct colours.*

Proof. The argument mirrors Lemma 3.4, but extends a colouring *downward* from G' rather than carrying one forward from H_1.

Since G is 4-colourable, by Tait's theorem $G' = \mathrm{dual}(G)$ admits a proper 3-edge-colouring ξ. Apply Lemma 2.4 to ξ at the pentagonal face F_v selected in step (1): the external vector $f = (f_0, \ldots, f_4)$ at F_v under ξ has the $(3,1,1)$ cyclic-consecutive shape, with a block of three consecutive positions $\{p, p+1, p+2\} \pmod 5$ holding a common colour c, and the remaining two positions $\{p+3, p+4\}$ holding the two non-c colours, one each. The algorithm's choice of i_1 forces $\{i_1+3, i_1+4\}$ inside the c-block (so the chord is consistently coloured) and the three positions $\{i_1, i_1+1, i_1+2\}$ pairwise distinct; in particular $i_1 + 1$ lies *outside* the c-block.

Choose φ_1 to be the proper 3-edge-colouring of H_1 that agrees with ξ on every surviving edge and assigns each new edge at A_j the unique third colour at A_j. Then $\varphi_1(\mathrm{spike}_1) = f_{i_1+1}$, a value not equal to c, while $\varphi_1(\mathrm{merged}_1) = f_{i_1+3} = c$, so $\varphi_1(\mathrm{spike}_1) \neq \varphi_1(\mathrm{merged}_1)$.

The same argument repeats at every step $k \geq 2$: the external vector at the chosen pentagonal face under φ_{k-1} has the $(3,1,1)$ cyclic-consecutive shape (Lemma 2.4), the algorithm's index choice i_k puts i_k+3, i_k+4 inside the colour block and i_k+1 outside, and step (3d) thus assigns $\varphi_k(\mathrm{spike}_k) \neq \varphi_k(\mathrm{merged}_k)$. The algorithm preserves these colours through every later step, so $\varphi_{t^*}(\mathrm{spike}_t) \neq \varphi_{t^*}(\mathrm{merged}_t)$ for every $t \in \{1, \ldots, t^*\}$. □

Conjecture 3.6. *Let G be a minimal counterexample to the Four Colour Theorem, and let $\widehat{G}'_{v,i}$ be a reduced dual of $G' = \mathrm{dual}(G)$. Then for every proper* 3*-edge-colouring φ of $\widehat{G}'_{v,i}$ there exist a face F of $\widehat{G}'_{v,i}$ and two distinct edges $e_1, e_2 \in \partial F$, with neither e_1 nor e_2 equal to the merged edge, such that*

(1) $\varphi(e_1) = \varphi(e_2)$, *and*
(2) e_1, e_2, *and the merged edge all lie on a common $\{a, b\}$-Kempe cycle of φ.*