THREE-COLOUR RESTRICTIONS FOR NESTED TIRE GRAPHS

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Abstract. We study three-colour boundary restrictions suggested by the nested tire decomposition of a plane triangulation. A level source induces a rooted tree of tire treads, and global colouring questions factor through local tread colourings together with compatibility along nested boundary cycles. We formulate a level-cycle three-colour restriction, exhibit counterexamples to two overly strong forms, and record exhaustive evidence for a surviving source-dependent conjecture. We also introduce seam language for minimum Four Colour Theorem counterexamples in the tire-tree framework.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

The structural study of such a minimal counterexample is the overarching motivation for the present line of work. The companion decomposition paper [3] establishes the foundational vocabulary — level sources, dual depth, tire graphs, medial tire graphs, and tire-tree decompositions — on which this paper builds. The companion dual paper [4] develops nested-cycle structure theorems and chain-pigeonhole conjectures for tire annular subgraphs of G'.

Related work. The structural object underlying this programme — the set of proper 4-colourings of a boundary cycle that extend to a colouring of a bounded planar region — is classical. Birkhoff's reducibility analysis of the diamond configuration [5] is the earliest instance of computing such extension sets to attack the Four Colour Theorem; the chromatic polynomial framework of Birkhoff and Lewis [6] systematised the counting. Tutte studied how the chromatic polynomial of a rooted planar triangulation decomposes along its outer boundary [9] and developed an algebraic theory of graph colourings organised around separating subgraphs [8, 7]. The most recent and structurally closest parallel is Dvořák and Lidický's analysis of *coloring count cones* [12], which characterises the possible boundary-extension functions on a fixed outer cycle of a near-triangulation. The Heesch–Appel–Haken approach [10, 11] also uses boundary-extension reasoning, but case-by-case on a finite unavoidable set of local configurations rather than as part of a global structural induction.

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The tire-tree decomposition used here differs from each of these in shape rather than ingredients. Birkhoff, Tutte, and Dvořák–Lidický all study *one* boundary; Heesch and the cleaned-up Appel–Haken proof [11] study a finite collection of local boundaries. The present framework organises the entire triangulation into a hierarchy of annular regions glued along level cycles, and asks whether boundary-extension constraints compose compatibly up the hierarchy. To the authors' knowledge, no prior work on the Four Colour Theorem has been organised around a global nested-cycle decomposition of this kind.

2. Background from nested tire decompositions

We use the terminology and structural results of [3]. In particular, a level source induces levels in a plane maximal planar graph, the depth-d inner-dual components determine tire graphs, and the resulting tire treads form a rooted tire tree $\mathcal{T}(G, S)$. For a tread T, we write $B^{(T)}_{\mathrm{out}}$ and $B^{(T)}_{\mathrm{in}}$ for its outer and inner boundary data, $O^{(T)}$ for its inner outerplanar graph, and G_T for the triangulated disk on the descendant side of $B^{(T)}_{\mathrm{out}}$. The base paper also proves that each tire tread has an outerplanar inner dual and that global colouring questions factor through local tread colourings together with compatibility along parent-child interfaces.

3. Single-tire colouring transfer

Theorem 3.1 (Tait correspondence: 4-colorings of a tire vs 3-edge-colorings of its inner dual)**.** *Let $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ be a tire graph (viewed as an annular triangulation of its tire tread R) and let Γ be its outerplanar inner dual, as supplied by* [3]*. Then*

$$\#\{\textit{proper 4-vertex-colorings of } T\}\big/|S_4| \;=\; \#\{\textit{proper 3-edge-colorings of } \Gamma\}\big/|S_3|.$$

That is, the number of 4-vertex-colorings of T up to permutation of the colour set $\{0, 1, 2, 3\}$ equals the number of 3-edge-colorings of Γ up to permutation of the colour set $\{1, 2, 3\}$.

Proof. The argument is the classical Tait correspondence [1] adapted to the annular triangulation T. Encode the four colours of a proper 4-vertex-coloring $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. For each interior annular edge e of T (whose two incident faces both lie in F_{ann}, contributing a Γ-edge e^*), set

$$\chi^*(e^*) \;:=\; c(u) + c(v) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \text{where } u, v \text{ are the endpoints of } e.$$

Since $c(u) \neq c(v)$, we have $\chi^*(e^*) \neq 00$, so χ^* takes values in $\{01, 10, 11\}$, which we identify with the 3-edge-coloring palette $\{1, 2, 3\}$.

Properness. At each Γ-vertex d_f corresponding to an annular triangle $f = \{u, v, w\}$, the three incident Γ-edges (one per cycle-edge of f) carry colours $c(u) + c(v),\ c(v) + c(w),\ c(u) + c(w)$. These three elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ sum to 0 and are pairwise distinct (their pairwise differences are $c(u) - c(w),\ c(v) - c(u),\ c(w) - c(v)$, each nonzero), so they form a permutation of $\{01, 10, 11\}$ — a proper edge colouring at d_f.

Surjectivity onto cosets. Given a proper 3-edge-coloring χ^* of Γ, the equation $c(u) + c(v) = \chi^*(e^*)$ admits exactly $|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$ solutions $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ (a global translation is the only freedom). Hence the map $c \mapsto \chi^*$ is 4-to-1.

Count. Therefore $\#\{4\text{-colorings of } T\} = 4 \cdot \#\{3\text{-edge-colorings of } \Gamma\}$. Dividing by $|S_4| = 24$ on the left and $|S_3| = 6$ on the right (since S_4 acts faithfully on the 4-colorings and S_3 on the 3-edge-colorings, and the 4-to-1 map respects the $S_4/S_3 \cong S_3$ quotient via the natural surjection $S_4 \twoheadrightarrow S_3$) gives the stated equality. □

Remark 3.2. Theorem 3.1 reduces the 4-colouring count of a tire to the 3-edge-coloring count of its outerplanar inner dual Γ. For the cycle case $\Gamma \cong C_{\mu+\nu}$ (the spoke-only case described in [3]), the cycle chromatic polynomial at 3 colours gives $2^{\mu+\nu}+2(-1)^{\mu+\nu}$. For an inner dual with one or more non-crossing chords, the count depends on the chord structure, not just on the pair (number of vertices, number of chords): two outerplanar graphs with the same number of vertices and number of chords can have different proper 3-edge-coloring counts depending on how the chords are arranged (nested, sequential, sharing vertices, etc.). Every such count can nevertheless be computed in linear time by tree-decomposition methods, since outerplanar graphs have treewidth at most 2 and the edge-chromatic polynomial admits a deletion–contraction recursion that respects the cycle-plus-chord structure.

Definition 3.3 (Boundary-state chromatic transfer)**.** Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph. Choose a cut along one annular edge if both boundaries are non-degenerate; in the degenerate case make no cut. The tread becomes a triangulated disk $\widetilde{R}$. Let

$$f_1, f_2, \ldots, f_m$$

be any shelling order of the triangular faces of $\widetilde{R}$, i.e. an order in which each initial union $\widetilde{R}_i := f_1 \cup \cdots \cup f_i$ is a disk. Such an order is obtained by taking an outerplanar embedding of the inner dual Γ from [3] and repeatedly removing an outer-face ear.

For each i, let A_i be the *frontier*: the vertices of T incident to at least one processed face in $\widetilde{R}_i$ and to at least one still-unprocessed constraint, where the unprocessed constraints are the remaining annular faces together with any edge of O not yet tested by the transfer. A *boundary state* on A_i is a partition π of A_i into colour classes, subject to the condition that adjacent vertices of $T[A_i]$ lie in distinct blocks. We write $r(\pi)$ for the number of blocks of π.

Theorem 3.4 (Chromatic polynomial of a tire by frontier transfer)**.** *For every tire graph T, the chromatic polynomial $P_T(q)$ is computed by the following boundary-state dynamic program.*

Initialize the table at $i = 0$ with the empty frontier state of weight 1*. When the next triangular face $f_i = \{x, y, z\}$ is attached, pass from states on A_{i-1} to states on A_i as follows.*

1. *Introduce any vertices of f_i not already present in A_{i-1}, assigning each such vertex either to an existing colour block not containing one of its already-coloured neighbours, or to a new block.*
2. *Reject every assignment in which two adjacent vertices of the triangle f_i lie in the same block. Also reject every assignment in which an edge of O whose two endpoints have now both appeared for the first time as a tested pair has both endpoints in the same block. Thus chords and bridges of the inner outerplanar graph are enforced exactly when their second endpoint becomes visible to the transfer.*
3. *Delete from the state every vertex no longer incident to an unprocessed constraint. If deleting a vertex removes the last representative of its colour*

block from the frontier, multiply that transition by 1*; the colour has already been chosen.*

(4) *If a new vertex is assigned to a new colour block while the current frontier state has r colour blocks, multiply that transition by $q-r$. If several new colour blocks are created in the same triangle, the factors are $(q-r)(q-r-1)\cdots$ in the order of creation.*

After f_m is processed, the frontier is empty. The single remaining weight is $P_T(q)$.

Proof. The construction is the standard transfer for the chromatic polynomial, specialized to the tire shelling. The frontier state records exactly the equality pattern among colours that can still affect unprocessed faces. Since colour names are irrelevant to the chromatic polynomial, states are quotiented by the natural action of the symmetric group on the colour set; a state with r visible colour blocks can be extended by a genuinely new colour in $q-r$ ways.

Each transition accounts for all proper colourings of the enlarged processed disk $\widetilde{R}_i$ that restrict to the resulting frontier state, and accounts for none that violate an edge of the newly attached triangle or untested edge of O. Vertices removed from the frontier have no future incident unprocessed constraint, so their actual colour names can no longer influence compatibility and may be forgotten. Induction on i therefore shows that the table after step i is precisely the orbit-count generating function for proper colourings of $\widetilde{R}_i$ by frontier state. At $i=m$ no vertices remain active, so the accumulated weight counts all proper colourings of T. Because the weights are polynomials in q, this count is the full chromatic polynomial. □

Remark 3.5 (Spoke-only transfer matrix)*.* In the spoke-only case with both boundaries simple cycles, the method has a particularly small form. Cut the annulus along one spoke and walk around the resulting strip. Each step adds one triangle sharing an edge with the previous processed strip, so the frontier consists of two or three consecutive boundary vertices. Up to colour permutation there are only the possible equality patterns among those active vertices, with adjacent vertices required to be distinct. The chromatic polynomial is therefore the trace of a finite transfer matrix whose entries are polynomials in q; the matrix depends only on the local triangle type encountered while walking around the tread. Chords or cut-vertices of O enlarge the frontier only at the corresponding outerplanar ears, and are handled by the same state rule of Theorem 3.4.

Remark 3.6 (Motivation for level-cycle restrictions)*.* The tire-tree decomposition reduces global colouring questions to local choices on treads together with compatibility along nested boundary cycles. Without further structure, the number of boundary colour states can grow quickly as one descends the tree: each seam or level cycle may in principle carry any proper restriction of a 4-colouring. The following restriction is meant to test whether this state space can be compressed. If level cycles can always be made to omit one colour, then each such interface behaves like a three-colour boundary object, while still allowing different cycles to omit different colours. This would not by itself solve the gluing problem, but it would give a simpler target class of boundary states for arguments about nested tire trees.

Definition 3.7 (Level-cycle three-colour restriction)**.** Let G be a maximal planar graph, let $S \subseteq V(G)$ be a level source, and let $c\colon V(G) \to \{1,2,3,4\}$ be a proper

4-vertex-colouring of G. We say that c has the *level-cycle three-colour restriction* with respect to S if, for every level $d \geq 0$ and every simple cycle $C \subseteq G[L_d]$, the colour set used on C has size at most three:

$$|c(V(C))| \leq 3.$$

Equivalently, every simple cycle contained in a single level omits at least one of the four colours. The omitted colour may depend on the cycle; in particular, distinct cycles in the same level, the same tire tread, or the same inner outerplanar component are not required to omit the same colour.

Conjecture 3.8 (False universal-source form)**.** *Let G be a maximal planar graph and let $S \subseteq V(G)$ be any level source. Then G admits a proper 4-vertex-colouring with the level-cycle three-colour restriction with respect to S.*

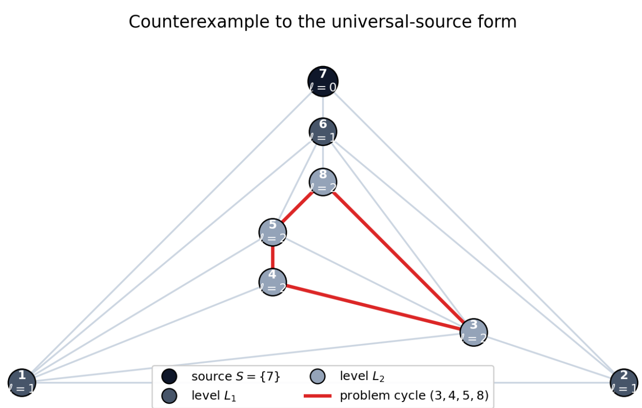

FIGURE 1. The 8-vertex counterexample to the universal-source form. With source $S = \{7\}$, the level cycle $(3, 4, 5, 8)$ lies in L_2 and forces all four colours in every proper 4-vertex-colouring.

Example 3.9 (Counterexample to Conjecture 3.8)**.** Let G be the maximal planar graph on vertex set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ with edge set

$$\begin{aligned} E(G) = \{&12, 13, 14, 15, 16, 17, 23, 26, 27, 34, 35, 36, 38,\\ &45, 56, 58, 67, 68\}. \end{aligned}$$

Here ij denotes the edge $\{i, j\}$. Take the vertex source $S = \{7\}$. The corresponding levels are

$$L_0 = \{7\}, \qquad L_1 = \{1, 2, 6\}, \qquad L_2 = \{3, 4, 5, 8\}.$$

Inside $G[L_2]$ the vertices $(3, 4, 5, 8)$ form a simple cycle. In every proper 4-vertex-colouring of G, these four vertices receive four distinct colours. The edges 34, 45, 58, 38, and 35 force all pairs among $\{3, 4, 5, 8\}$ except possibly $\{4, 8\}$ to have distinct colours. If 4 and 8 had the same colour, then vertex 6, which is adjacent to 3, 5, and 8, would have to use the fourth colour; but vertex 1 is adjacent to 3, 4, 5, and 6, and would then be adjacent to all four colours, impossible in a proper 4-colouring. Hence 4 and 8 also have distinct colours, so the level cycle $(3, 4, 5, 8)$ uses all four colours in every proper 4-colouring of G. Therefore no proper 4-colouring has the level-cycle three-colour restriction with respect to $S = \{7\}$.

An inner-boundary refinement. The level-cycle restriction constrains *every* simple cycle in every level. For the tire-tree program, the cycles that actually carry boundary state are fewer: each tire transfers colour information across its tread between its two boundaries, so it is the tire *inner boundaries* $B_{\mathrm{in}}^{(T)}$ — not all level cycles — that one wishes to compress. This motivates a restriction stated directly in the objects of the decomposition.

Definition 3.10 (Tire inner-boundary three-colour restriction)**.** Let G be a maximal planar graph, let $v_0 \in V(G)$ be a vertex source on the outer face of Π_G, and let $c\colon V(G) \to \{1,2,3,4\}$ be a proper 4-vertex-colouring of G. We say c has the *tire inner-boundary three-colour restriction* with respect to $\mathcal{T}(G,\{v_0\})$ if every tire tread $T \in \mathcal{T}(G,\{v_0\})$ satisfies

$$|c(V(B_{\mathrm{in}}^{(T)}))| \leq 3,$$

i.e. the inner boundary of every tire omits at least one of the four colours. (A degenerate inner boundary is a single vertex and the condition is then vacuous.)

Conjecture 3.11 (Tire inner-boundary three-colour conjecture)**.** *Every maximal planar graph G admits a vertex source $v_0 \in V(G)$ and a proper 4-vertex-colouring c of G such that c has the tire inner-boundary three-colour restriction with respect to $\mathcal{T}(G,\{v_0\})$.*

A counterexample at $n = 14$. Conjecture 3.11 is in fact false. An exhaustive search over the triangulations enumerated by `plantri` at $n = 14$ encounters a graph $G^\star$ on 14 vertices and 36 edges — specifically, the graph at index 263993 in the `plantri` enumeration — for which no vertex source admits any witness.

Example 3.12 (Counterexample to Conjecture 3.11)**.** Let $G^\star$ be the maximal planar graph with vertex set $\{1,2,\dots,14\}$ and edge set

$$\begin{aligned} E(G^\star) = \{&12, 13, 14, 15, 16, 17, 18,\\ &23, 24, 26, 28, 29, 2\,10,\\ &34, 45, 46, 4\,10, 56, 67, 69, 6\,10,\\ &78, 79, 7\,11, 7\,12, 7\,13,\\ &89, 8\,12, 8\,13, 8\,14,\\ &9\,11, 9\,12, 9\,14,\\ &11\,12, 12\,13, 12\,14\}.\end{aligned}$$

The graph $G^\star$ is a 3-connected (but not 5-connected) planar triangulation with degree sequence $(7,7,7,7,7,7,6,6,3,3,3,3,3,3)$ and exactly 96 proper 4-vertex colourings. For *every* choice of vertex source $v_0 \in V(G^\star)$, each of the 96 proper 4-colourings of $G^\star$ has some tire whose inner boundary uses all four colours. A planar embedding is shown in Figure 2.

The failure was verified by enumerating, for each of the 14 vertex sources, all 96 proper 4-colourings of $G^\star$ and computing the inner boundary $V(B_{\mathrm{in}}^{(T)})$ of every tire T as the level-$(d+1)$ vertices of the corresponding depth-d dual component. Each source has exactly two non-degenerate inner boundaries (size ≥ 4), and every proper 4-colouring assigns all four colours to at least one of them.

The graph $G^\star$ does not refute Conjecture 3.13: the vertex source $v_0 = 10$ admits a proper 4-colouring under which every simple level cycle uses at most three colours.

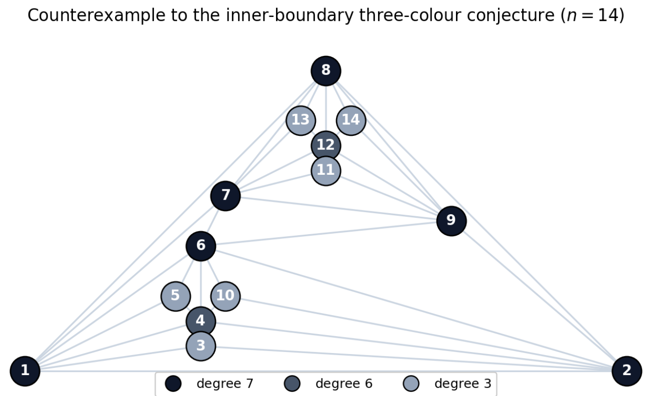

FIGURE 2. The 14-vertex counterexample $G^\star$ to Conjecture 3.11 in a planar embedding. The six degree-3 vertices split into two triples, $\{3, 5, 10\}$ each adjacent to a triangle in the core $\{1, 2, 4, 6\}$, and $\{11, 13, 14\}$ each adjacent to a triangle in the core $\{7, 8, 9, 12\}$; the two cores are joined by the edges $17, 28, 69$ together with 12.

The surviving level-cycle conjecture.

Conjecture 3.13 (Level-cycle three-colour conjecture)**.** *Let G be a maximal planar graph. Then there exists a level source $S \subseteq V(G)$ such that G admits a proper 4-vertex-colouring with the level-cycle three-colour restriction with respect to S.*

Enumeration for small n. We exhaustively enumerated all plane triangulation isomorphism classes with $4 \le n \le 13$ vertices and searched the vertex sources for each graph. No counterexample to Conjecture 3.13 appeared in this range. Table 1 records the size of the search space and the number of triangulations that admit a witness.

n	triangulations	with witness
4	1	1
5	1	1
6	2	2
7	5	5
8	14	14
9	50	50
10	233	233
11	1249	1249
12	7595	7595
13	49566	49566

TABLE 1. Exhaustive vertex-source search for the level-cycle three-colour conjecture on all triangulation isomorphism classes with $4 \le n \le 13$. Every triangulation in this range admits at least one vertex source witnessing the conjecture.

We also tested the six dual triangulations of the Holton–McKay graphs, which lie just beyond this census, and found witnesses in each case.

The 5-connected slice at $n \le 24$. As a compact test above the full small-n census, we also enumerated the 5-connected triangulations at $14 \le n \le 24$ with `plantri -c5 -a`. These are especially rigid triangulations, and the slice remains small enough to check exhaustively. Every graph in this slice admits a vertex source witnessing Conjecture 3.13.

n	5-connected triangulations	with witness
14	1	1
15	1	1
16	3	3
17	4	4
18	12	12
19	23	23
20	71	71
21	187	187
22	627	627
23	1970	1970
24	6833	6833

TABLE 2. The 5-connected triangulations at $14 \le n \le 24$ generated by `plantri -c5 -a`. All 9732 graphs in this slice admit a vertex source witnessing the level-cycle three-colour conjecture.

Definition 3.14 (Seam)**.** A *seam* of a maximal planar graph G is a simple cycle $C \subset G$ such that, for some vertex $v_0 \in V(G)$, $C = B_{\text{out}}^{(T)}$ for some non-root tread T in $\mathcal{T}(G, \{v_0\})$.

By the tire-tree decomposition theorem of [3], every seam C separates G into:

- the *seam interior* G_T, the triangulated disk on the T-descendant side of C;
- the *seam exterior* $G_C^{\text{ext}} := G \setminus \text{int}(G_T)$, the triangulated polygon with outer face bounded by C on the side containing v_0;

both sharing C. A seam is *non-trivial* if both $V(G_T) \setminus V(C)$ and $V(G_C^{\text{ext}}) \setminus V(C)$ are non-empty.

For any seam C and either side $X \in \{G_T, G_C^{\text{ext}}\}$, write

$$\text{Col}(X \mid C) \;:=\; \left\{ c|_{V(C)} \;:\; c \text{ a proper 4-colouring of } X \right\} \;\subseteq\; \{1,2,3,4\}^{V(C)}$$

for the set of C-restricted 4-colourings induced by 4-colourings of X (each element is a proper 4-colouring of the cycle C).

Definition 3.15 (Partial tire tree)**.** Let T_r be a tire tread in $\mathcal{T}(G, S)$ with outer boundary cycle $C_{T_r} = B_{\text{out}}^{(T_r)}$, and let G_{T_r} be the triangulated disk inside C_{T_r} given by the tire-tree decomposition theorem of [3]. The *partial tire tree* with root T_r, written $G_{T_r}^{\circ}$, is the induced subgraph of G on the vertex set $V(G_{T_r}) \setminus V(C_{T_r})$ — i.e. G_{T_r} with the seam-cycle vertices removed.

Equivalently, $V(G_{T_r}^{\circ})$ is the set of vertices of G strictly inside C_{T_r} on the side away from the level source, and $E(G_{T_r}^{\circ})$ consists of the edges of G both of whose endpoints lie in this strict interior. The tree-of-tire-treads structure of $G_{T_r}^{\circ}$ is the sub-tree of $\mathcal{T}(G, S)$ rooted at T_r, with T_r's outer boundary peeled away.

Lemma 3.16 (Seam edges are shared by at most one other depth-d seam)**.** *Let G be a maximal planar graph with single-vertex level source $S = \{v_0\}$, fix $d \ge 1$,* *and let $e \in E(G)$ be an edge lying on the seam $C_T = B^{(T)}_{\text{out}}$ of some tire tread $T \in \mathcal{T}(G,S)$ at depth d. Then there is at most one other tire tread $T' \in \mathcal{T}(G,S)$ at the same depth d with $e \in C_{T'}$.*

Proof. By the child–face correspondence in the tire-tree construction of [3], C_T is the boundary cycle of a bounded face of the parent's inner outerplanar graph $O^{(T_p)}$, where $T_p \in \mathcal{T}(G,S)$ is the parent of T at depth $d-1$. The inner dual of a connected outerplanar graph is a tree, so each edge of $O^{(T_p)}$ lies on at most two of its bounded face cycles. Hence e lies on at most one other bounded face cycle of $O^{(T_p)}$, corresponding, by the same child–face bijection, to at most one sibling of T at depth d whose seam contains e. □

Conjecture 3.17 (Seam structure of minimum 4CT counterexamples, sketch)**.** *Suppose the Four Colour Theorem fails: there exists a maximal planar graph that is not* 4*-colourable. Let G be a* minimum *such counterexample (with $|V(G)|$ minimal among non-*4*-colourable maximal planar graphs). Then:*

Restatement-of-classical content.

- (C1) Bilateral colourability. *For every non-trivial seam C of G, both* $\mathrm{Col}(G_T \mid C)$ *and* $\mathrm{Col}(G^{\text{ext}}_C \mid C)$ *are non-empty.*
- (C2) Bilateral incompatibility. *For every non-trivial seam C,*

$$\mathrm{Col}(G_T \mid C) \;\cap\; \mathrm{Col}(G^{\text{ext}}_C \mid C) \;=\; \emptyset.$$

- (C3) Length lower bound (Birkhoff). *Every non-trivial seam C of G has $|V(C)| \geq 6$.*

*(C1) and (C2) together restate "G is a counterexample whose every internal cut by a seam splits into two colourable pieces with incompatible boundary palettes"; (C1) follows from minimality applied to each side after closing the polygonal outer face by a single apex, (C2) from G itself being non-*4*-colourable. (C3) is Birkhoff's internally-*6*-connected condition restated in the seam language.*

Substantive (speculative) content.

- (C4) Innermost obstruction. *There exists a vertex source $v_0 \in V(G)$ and a* leaf *tread $T^* \in \mathcal{T}(G,\{v_0\})$ (a tread with no children in the tree-of-treads) such that:*
  - (i) *the seam interior G_{T^*} is, up to plane iso, one of a finite list of* minimal seam configurations*, characterized by their boundary palette* $\mathrm{Col}(G_{T^*} \mid C_{T^*})$ *being a specific proper subset of the proper* 4*-colourings of the cycle C_{T^*};*
  - (ii) *the path in $\mathcal{T}(G,\{v_0\})$ from the root T_0 to T^* is an* obstruction chain*:* $\mathrm{Col}(G_T \mid C_T)$ *is monotonically restricted (under the natural pull-back along parent–child seams of the parent–child seam pull-back described in* [3]*) as T descends from the root to T^*, with the final restriction at T^* being incompatible with the v_0-side palette.*

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