# COLORING NESTED TIRE DUAL GRAPHS

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Abstract. This is a follow-up to [2], which establishes the basic vocabulary of tire graphs $T$ and dual depth, along with the tire-component lemma and the tire-tread partition theorem. Building on those structural results, we define the *partial tire dual* $D(T)$ and analyse its structure in the spoke-only case (a corona graph $C_{n+m} \circ K_1$), give an edge-vertex coloring bijection that reduces counting proper 3-edge-colorings of $D(T)$ to counting proper 3-vertex-colorings of a cycle, and develop the tire-annular-subgraph, face-connector, and inner/outer-spoke structures in $G'$. A concluding section records a Latin-substructure conjecture for chain-pigeonhole compatibility of adjacent tires.

## 1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation $G$ is properly 4-vertex-colourable if and only if its dual cubic graph $G'$ is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper is the second in a series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual $G'$ and its dual depth, and tire graphs — is developed in the companion paper [2]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source* $S$ and $G$-vertex levels $\ell_G(v)$;
- the inner planar dual $G'$ ([2, Definition 1.3]);
- *dual depth* $\delta_G(d_f)$ ([2, Definition 1.4]);
- *tire graph* $T = (B_{\text{out}}, O, E_{\text{ann}})$ with outer/inner boundaries and annular edges ([2, Definition 1.5]);
- face/edge counts ([2, Remark 1.6]);
- the *tire-component lemma* ([2, Lemma 1.8]), which exhibits each connected component of $G'_d$ as a tire graph whose tire tread is the union of its depth-$d$ faces;
- the *tire-tread partition theorem* ([2, Theorem 1.9]), which shows the tire treads from a level source partition the bounded faces of $G$.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding $\Pi_G$. We write $|V| = n$, so $|E| = 3n - 6$ and $G$ has $2n - 4$ triangular faces.

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**Definition 1.1** (Partial tire dual)**.** Let $T = (B_{\rm out}, O, E_{\rm ann})$ be a tire graph in the sense of [2, Definition 1.5], and let $F_{\rm ann}$ denote the set of triangular faces of $T$ in the tire tread (the closed region between $B_{\rm out}$ and $B_{\rm in}$). The *partial tire dual* of $T$, written $D(T)$, is the graph defined as follows.

*Vertices.*

(V1) For each face $f \in F_{\rm ann}$, an *interior vertex* $d_f$ of $D(T)$.
(V2) For each edge $e \in E(B_{\rm out})$, a *leaf vertex* $\ell_e^{\rm out}$.
(V3) For each occurrence of an edge in the closed walk $B_{\rm in}$ (= the outer-face boundary walk of $O$), a *leaf vertex* $\ell_e^{\rm in}$. (When $O$ is 2-connected each edge appears once; cut-vertices and bridges of $O$ may cause an edge or vertex to appear more than once.)

*Edges.*

(E1) For each edge $e \in E(T)$ whose two incident faces both lie in $F_{\rm ann}$ (an *interior annular edge*), one edge $\{d_{f_1}, d_{f_2}\} \in E(D(T))$ where $f_1, f_2 \in F_{\rm ann}$ are the two annular faces incident to $e$.
(E2) For each $e \in E(B_{\rm out})$, one edge $\{d_f, \ell_e^{\rm out}\} \in E(D(T))$ where $f \in F_{\rm ann}$ is the unique annular face incident to $e$. The leaf $\ell_e^{\rm out}$ has degree 1.
(E3) For each occurrence of $e$ on the boundary walk $B_{\rm in}$, one edge $\{d_f, \ell_e^{\rm in}\} \in E(D(T))$ where $f \in F_{\rm ann}$ is the annular face incident to $e$ on the side of that occurrence. The leaf $\ell_e^{\rm in}$ has degree 1.

**Proposition 1.2** (Structure of $D(T)$ when the annular triangulation is spoke-only)**.** *Suppose $B_{\rm out}$ is a simple cycle of length $n$, $O$ is a 2-connected outerplanar graph whose outer-face cycle $B_{\rm in}$ has length $m$, and $E_{\rm ann}$ consists only of* spokes *(edges with one endpoint in $V(B_{\rm out})$ and one in $V(B_{\rm in})$). Then each face $f \in F_{\rm ann}$ has exactly one boundary edge (on $B_{\rm out}$ or $B_{\rm in}$) and two interior annular edges, and consequently $D(T)$ is isomorphic to the corona graph $C_{n+m} \circ K_1$: a cycle of length $n+m$ on the interior vertices $\{d_f\}$, with one leaf attached to each cycle vertex.*

*In particular, $|V(D(T))| = 2(n+m)$ and $|E(D(T))| = 2(n+m)$.*

*Proof.* Each annular triangle $f$ in a spoke-only triangulation has the form $\{x, y, z\}$ with $x \in V(B_{\rm out})$, $y \in V(B_{\rm in})$, and $z$ also in $V(B_{\rm out}) \cup V(B_{\rm in})$. Of its three edges, the one between the two same-side vertices ($x$-$z$ if both on $B_{\rm out}$, or $y$-$z$ if both on $B_{\rm in}$) is a boundary edge of the tire tread; the other two edges are spokes.

So each $d_f$ has degree 3 in $D(T)$: two from interior edges (= spokes shared with adjacent annular faces) and one leaf. The induced subgraph on $\{d_f : f \in F_{\rm ann}\}$ is 2-regular; together with the connectedness of the tire tread this forces it to be a single cycle. By [2, Remark 1.6], the cycle has length $n+m$, and there are also $n+m$ leaves attached one-per-cycle-vertex. □

**Proposition 1.3** (Edge–vertex coloring bijection for $D(T)$)**.** *Let $T$ be a tire graph satisfying the spoke-only hypothesis of Proposition 1.2 (so $D(T) \cong C_{n+m} \circ K_1$). Let $\Gamma \cong C_{n+m}$ be the interior dual subgraph of $D(T)$ induced on the interior dual vertices $\{d_f : f \in F_{\rm ann}\}$. Then the number of proper 3-edge-colorings of $D(T)$ equals the number of proper 3-vertex-colorings of $\Gamma$, both given by*

$$2^{n+m} + 2 \cdot (-1)^{n+m}.$$

*Proof.* Write $L = n + m$, $\Gamma = C_L$. We construct mutually inverse bijections.

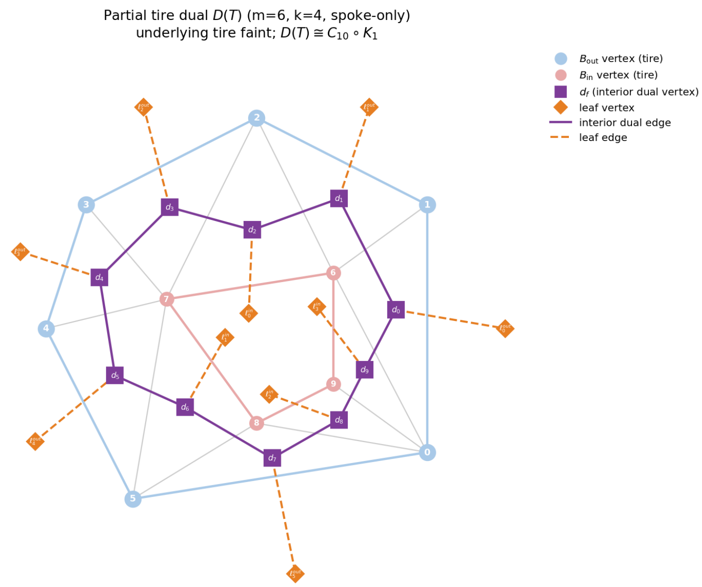


FIGURE 1. The partial tire dual $D(T)$ (purple squares + orange diamonds) drawn on top of a small tire graph $T$ (faint) with $m = 6$ and $k = 4$. The ten interior vertices $d_f$ at the centroids of the annular triangles form a single 10-cycle (solid purple); each boundary edge of the tire tread (either of $B_{\text{out}}$ or of $B_{\text{in}}$) contributes a degree-1 leaf (orange diamond) attached to the unique annular face incident to it (dashed orange), giving the structure $C_{10} \circ K_1$ of Proposition 1.2.

*Step 1: proper* 3*-edge-colorings of* $D(T) \leftrightarrow$ *proper* 3*-edge-colorings of* $C_L$. Given a proper 3-edge-coloring $\chi$ of $D(T)$, the three edges incident to any $d_f$ carry three distinct colors; in particular the two cycle edges incident to $d_f$ carry distinct colors, so $\chi|_{E(C_L)}$ is a proper 3-edge-coloring of $C_L$. Conversely, given a proper 3-edge-coloring $\psi$ of $C_L$, the two cycle edges at any $d_f$ have distinct colors, so a unique third color is available; assign that color to $d_f$'s leaf edge. The resulting extension to $D(T)$ is proper at every $d_f$ and vacuously proper at every leaf (degree 1), and the two maps are inverse to each other. Therefore

$$\#\{\text{proper 3-edge-colorings of } D(T)\} = \#\{\text{proper 3-edge-colorings of } C_L\}.$$

*Step 2: proper* 3*-edge-colorings of* $C_L \leftrightarrow$ *proper* 3*-vertex-colorings of* $L(C_L) \cong C_L$. The line graph $L(C_L)$ of a cycle of length $L$ is again a cycle of length $L$; proper edge-colorings of $C_L$ are by definition proper vertex-colorings of $L(C_L)$.

*Step 3: count.* The chromatic polynomial of the cycle is $P(C_L, k) = (k-1)^L + (-1)^L(k-1)$; at $k = 3$ this gives $2^L + 2 \cdot (-1)^L$. □

*Remark* 1.4. Proposition 1.3 reduces counting proper 3-edge-colorings of $D(T)$ to counting proper 3-vertex-colorings of a single cycle, giving a closed form $2^{n+m} + 2(-1)^{n+m}$ that depends only on $n + m$ (not on the specific spoke-only annular triangulation, nor on the chord structure of $O$). The count is preserved under the

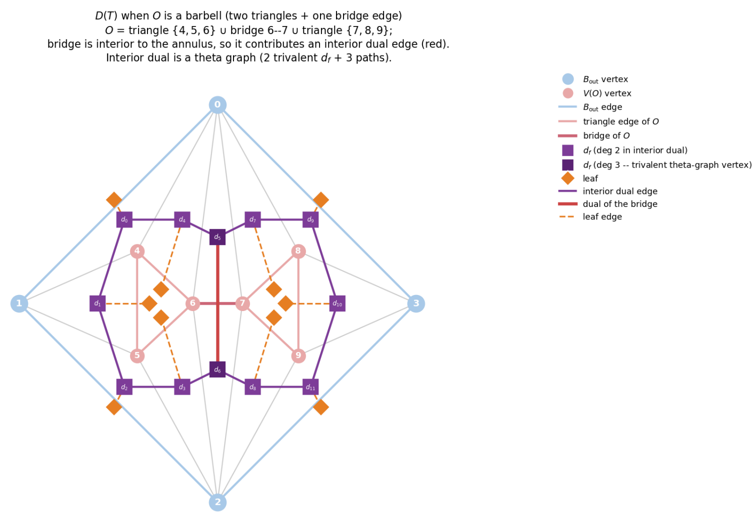


FIGURE 2. Partial tire dual $D(T)$ when the inner outerplanar graph $O$ has a bridge — here a non-trivial edge cut connecting two disjoint triangles. $B_{\text{out}}$ is a 4-cycle on $\{0, 1, 2, 3\}$ and $O$ is the barbell: triangle $\{4, 5, 6\}$ together with triangle $\{7, 8, 9\}$ joined by the bridge edge 6–7 (removing the bridge disconnects $O$). Because both faces incident to the bridge are annular triangles, the bridge contributes an *interior dual edge* (highlighted in red) rather than two leaves; consequently the interior dual subgraph is no longer the single $(n + m)$-cycle of Proposition 1.2, but a theta graph: the two trivalent vertices $d_5, d_6$ (the bridge-incident annular faces) are joined by three internally vertex-disjoint paths in $D(T)$. Leaves come only from $B_{\text{out}}$ ($n = 4$ leaves) and the six non-bridge edges of $O$ ($m_\partial = 6$ leaves, three for each triangle).

corona-with-$K_1$ structure of Proposition 1.2 precisely because each degree-1 leaf imposes no proper-edge-coloring constraint on itself; its color is freely determined as the missing third color at its attached interior vertex.

**Definition 1.5** (Tire annular subgraph)**.** Let $G$ be a maximal planar graph with embedding $\Pi_G$ and inner planar dual $G'$ (as in [2, Definition 1.3] above). Let $T = (B_{\text{out}}, O, E_{\text{ann}}) \subseteq G$ be a tire graph ([2, Definition 1.5]), and let $F_{\text{ann}} \subseteq F(G)$ denote its set of annular faces. The *tire annular subgraph* of $T$ in $G'$ is

$$T'_{\text{ann}} \;:=\; G'\big[\,\{d_f : f \in F_{\text{ann}}\}\,\big],$$

the subgraph of $G'$ induced on the dual vertices corresponding to the annular faces of $T$. We equip $T'_{\text{ann}}$ with the planar embedding inherited from $G'$ (which, by deletion of vertices outside the annulus, remains a planar embedding of $T'_{\text{ann}}$ in the sense of $\Pi_G$).

**Definition 1.6** (Tire annular face connector)**.** With $G, G', T$ as in Definition 1.5, let $f'$ be a face of the tire annular subgraph $T'_{\text{ann}}$ in its inherited embedding, and let $V(f') \subseteq V(T'_{\text{ann}})$ denote the set of vertices on the boundary walk of $f'$. The *tire* *annular face connector at* $f'$ is the subgraph

$$T'_{f'} \;:=\; \bigl(V(f') \cup N_{G'}(V(f'))\,,\ \{\, e \in E(G') : e \text{ is incident to } V(f')\,\}\bigr) \;\subseteq\; G',$$

i.e. the subgraph of $G'$ on the closed $G'$-neighborhood of $V(f')$ together with every $G'$-edge incident to $V(f')$.

**Definition 1.7** (Inner and outer spokes)**.** With $T'_{f'}$ as in Definition 1.6, regard $f'$ as an open region of $|\Pi_G|$ and write $\overline{f'}$ for its closure. The vertices of $V(T'_{f'}) \setminus V(f')$ lie in $|\Pi_G| \setminus \overline{f'}$ or in $f'$ (never on $\partial f'$, since the boundary walk of $f'$ is by definition the set $V(f')$). Partition

$$V(T'_{f'}) \setminus V(f') \;=\; V_{\text{out}}(T'_{f'}) \sqcup V_{\text{in}}(T'_{f'})$$

where

$$V_{\text{out}}(T'_{f'}) := \{\, v \in V(T'_{f'}) \setminus V(f') \;:\; v \notin \overline{f'}\,\},$$
$$V_{\text{in}}(T'_{f'}) := \{\, v \in V(T'_{f'}) \setminus V(f') \;:\; v \in f'\,\}.$$

The elements of $V_{\text{out}}(T'_{f'})$ are the *outer spokes* of $T'_{f'}$ (vertices not in $V(f')$ that lie outside the region bounded by $f'$); the elements of $V_{\text{in}}(T'_{f'})$ are the *inner spokes* of $T'_{f'}$ (vertices not in $V(f')$ that lie inside the region bounded by $f'$).

*Remark* 1.8. In the spoke-only setting of Proposition 1.2, the tire annular subgraph is $T'_{\text{ann}} = \Gamma \cong C_{n+m}$ (Proposition 1.3). This cycle has exactly two faces in its inherited embedding – one on each side of the cycle in $\Pi_G$ – and both face boundaries traverse all $n+m$ vertices, so $V(f') = V(\Gamma)$ for either choice of $f'$. Each interior dual vertex $d_f$ has $G'$-degree 3 (since $G$ is a triangulation), of which two edges lie in $\Gamma$ (cycle edges) and one edge points to a single non-annular face of $G$. Consequently $T'_{f'}$ has $n+m$ interior vertices plus the non-annular face vertices to which they connect, and is independent of the choice of $f'$. When $G$ consists only of the tire $T$ together with one source-side face inside $B_{\text{out}}$ and one $O$-side face inside $B_{\text{in}}$, $T'_{f'}$ recovers the planar dual of $T$ itself.

## 2. A conjectural Latin-style substructure

Empirical enumeration (notes `tire_fiber_data.tex`, `tire_fiber_chords.tex`, `tire_fiber_step2.tex`, `tire_fiber_step2_large.tex`) of edge 3-coloring distributions on the tire annular face connector $T'_{f'}$ across 46 adjacent-tire pairs at $|\gamma| \in \{3,4,5,6,9,12\}$ suggests that the chain-pigeonhole step on a shared cycle always succeeds. The data points to a structural mechanism: every edge-3-colourable tire admits at least one “Latin-flavoured” boundary configuration, and adjacent tires share this same substructure on their common cycle.

Concretely, fix a tire $T$ with inner outerplanar graph $O$ on $V(B_{\text{in}})$ and let $F(O)$ be the set of $O$-faces (in the tire's plane embedding, not counting the outer face $B_{\text{in}}$). For each $O$-face $f \in F(O)$, let $E_{\text{in}}(f) \subseteq E(B_{\text{in}})$ denote the set of $B_{\text{in}}$ edges on $f$'s boundary. In the Steiner-poor surrounding triangulation (where each $O$-face is a single face of $G$ and dualises to a single $G'$-vertex of degree $|E_{\text{in}}(f)|$ in $T'_{f'}$), proper edge 3-colouring of $T'_{f'}$ requires every $O$-face to have $|E_{\text{in}}(f)| \le 3$.

Let $\sigma_{B_{\text{in}}}$ denote the spoke colouring restricted to the $|V(B_{\text{in}})|$ inner-direction spoke positions on the dual annular cycle (equivalently: $\sigma$ indexed by $E(B_{\text{in}})$).

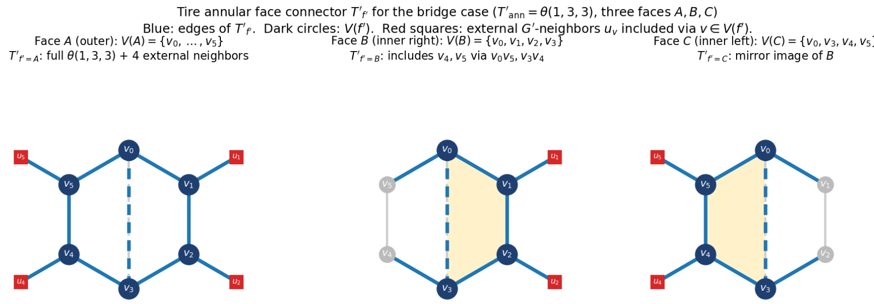


FIGURE 3. The bridge case: $T'_{\mathrm{ann}} = \theta(1,3,3)$ has three faces $A, B, C$ in its inherited embedding, with respective vertex sets $V(A) = \{v_0, \ldots, v_5\}$, $V(B) = \{v_0, v_1, v_2, v_3\}$, and $V(C) = \{v_0, v_3, v_4, v_5\}$. In the surrounding maximal planar $G$, the chord endpoints $v_0, v_3$ (the two annular faces sharing the bridge edge) have all three $G'$-edges inside $T'_{\mathrm{ann}}$, while each non-chord vertex $v_i$ ($i \in \{1,2,4,5\}$) contributes one $G'$-edge to an external non-annular neighbor $u_i$. Each panel highlights $T'_{f'}$ (blue) inside $G'$: dark circles are $V(f')$, gray circles are $G'$-neighbors of $V(f')$ within $T'_{\mathrm{ann}}$, and red squares are external $G'$-neighbors $u_i$. The choice of face $f'$ controls which external neighbors $u_i$ are pulled into $T'_{f'}$ (face $A$ pulls in all four; face $B$ pulls in $u_1, u_2$ and face $C$ pulls in $u_4, u_5$).

Define the *Latin-flavoured set* on $\gamma = B_{\mathrm{in}}$ as

$$\mathcal{L}(B_{\mathrm{in}}, O) := \left\{ \sigma : E(B_{\mathrm{in}}) \to \{1,2,3\} \;\middle|\; \sigma|_{E_{\mathrm{in}}(f)} \text{ is a permutation of } \{1,2,3\} \text{ for every } f \in F(O) \right\}.$$

That is, on every $O$-face's $B_{\mathrm{in}}$-edge boundary, all three colours appear exactly once (forcing $|E_{\mathrm{in}}(f)| = 3$ for each face — the maximally constrained case).

**Conjecture 2.1** (Latin-substructure conjecture)**.** *For any Steiner-poor edge-3-colourable tire $T$ with inner outerplanar graph $O$ such that every $O$-face has exactly 3 $B_{\mathrm{in}}$-edges, the realisable inner-spoke projection $\pi_D(\mathcal{C}(T'_{f'}))$ contains $\mathcal{L}(B_{\mathrm{in}}, O)$ as a subset. Moreover, $\mathcal{L}(B_{\mathrm{in}}, O)$ is invariant under the $S_3$ action on colours and has size at least $3! = 6$.*

**Conjecture 2.2** (Chain-pigeonhole compatibility from Latin substructure)**.** *Adjacent tires $T_1, T_2$ sharing a cycle $\gamma$ admit a joint edge 3-colouring whenever their respective inner-outerplanar structures $O^{(1)}, O^{(2)}$ both satisfy Conjecture 2.1. Equivalently: $\pi_D^{(1)}(\mathcal{C}(T_1)) \cap \pi_U^{(2)}(\mathcal{C}(T_2)) \supseteq \mathcal{L}(\gamma, O^{(1)}) \cap \mathcal{L}(\gamma, O^{(2)})$, and this last intersection is non-empty whenever the two face partitions of $E(\gamma)$ induced by $O^{(1)}, O^{(2)}$ share a common "Latin completion."*

The structural origin of these conjectures is the empirical observation that the smallest tested intersections on $\gamma$ are always exactly the 3! permutations of a single canonical pattern in which each $O$-face on $\gamma$'s side receives a permutation of $\{1,2,3\}$. For example, at $|\gamma| = 12$ with $O^{(1)}$ given by the chord matching $\{(0,3),(4,7),(8,11)\}$ (face structure $\{0,1,2\} \sqcup \{4,5,6\} \sqcup \{8,9,10\} \sqcup \{3,7,11\}$), the canonical pattern $(1, 2, 3, 2, 2, 1, 3, 3, 2, 3, 1, 1)$ assigns the permutation $(1, 2, 3)$ to the first face, $(2, 1, 3)$ to the second, $(2, 3, 1)$ to the third, and $(2, 3, 1)$ to the fourth. Every face receives all three colours.

A proof of Conjecture 2.1 would convert the chain-pigeonhole compatibility step into a structural theorem on $T'_{f'}$: it is not the rough abundance of valid spoke configurations that lets adjacent tires meet, but a specific Latin-square-flavoured substructure dictated by the face partition of each tire's inner outerplanar graph. See `notes/tire_fiber_step2_large.tex` for the data underlying this conjecture and `experiments/tire_fiber_counterexample_search.log` for the ongoing automated search.

## References

[1] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.
[2] E. Bauerfeld, *Nested Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.