COLORING NESTED TIRE DUAL GRAPHS

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Abstract. This is a follow-up to [2], which establishes the basic vocabulary of tire graphs T and dual depth. Building on those definitions, we define the *partial tire dual* $D(T)$ and analyse its structure in the spoke-only case (a corona graph $C_{n+m} \circ K_1$), prove the tire-component lemma exhibiting every BFS-level component as a tire graph, give an edge-vertex coloring bijection that reduces counting proper 3-edge-colorings of $D(T)$ to counting proper 3-vertex-colorings of a cycle, and develop the tire-annular-subgraph, face-connector, and inner/outer-spoke structures in G'. A concluding section records a Latin-substructure conjecture for chain-pigeonhole compatibility of adjacent tires.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper is the second in a series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual G' and its dual depth, and tire graphs — is developed in the companion paper [2]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source* S and G-vertex levels $\ell_G(v)$;
- the inner planar dual G' ([2, Definition 1.3]);
- *dual depth* $\delta_G(d_f)$ ([2, Definition 1.4]);
- *tire graph* $T = (B_{\text{out}}, O, E_{\text{ann}})$ with outer/inner boundaries and annular edges ([2, Definition 1.5]);
- face/edge counts ([2, Remark 1.6]).

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Partial tire dual). Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph in the sense of [2, Definition 1.5], and let F_{ann} denote the set of triangular faces of T in the tire tread (the closed region between B_{out} and B_{in}). The *partial tire dual* of T, written $D(T)$, is the graph defined as follows.

Vertices.

(V1) For each face $f \in F_{\text{ann}}$, an *interior vertex* d_f of $D(T)$.

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(V2) For each edge $e \in E(B_{\text{out}})$, a *leaf vertex* ℓ_e^{out}.

(V3) For each occurrence of an edge in the closed walk B_{in} (= the outer-face boundary walk of O), a *leaf vertex* ℓ_e^{in}. (When O is 2-connected each edge appears once; cut-vertices and bridges of O may cause an edge or vertex to appear more than once.)

Edges.

(E1) For each edge $e \in E(T)$ whose two incident faces both lie in F_{ann} (an *interior annular edge*), one edge $\{d_{f_1}, d_{f_2}\} \in E(D(T))$ where $f_1, f_2 \in F_{\text{ann}}$ are the two annular faces incident to e.

(E2) For each $e \in E(B_{\text{out}})$, one edge $\{d_f, \ell_e^{\text{out}}\} \in E(D(T))$ where $f \in F_{\text{ann}}$ is the unique annular face incident to e. The leaf ℓ_e^{out} has degree 1.

(E3) For each occurrence of e on the boundary walk B_{in}, one edge $\{d_f, \ell_e^{\text{in}}\} \in E(D(T))$ where $f \in F_{\text{ann}}$ is the annular face incident to e on the side of that occurrence. The leaf ℓ_e^{in} has degree 1.

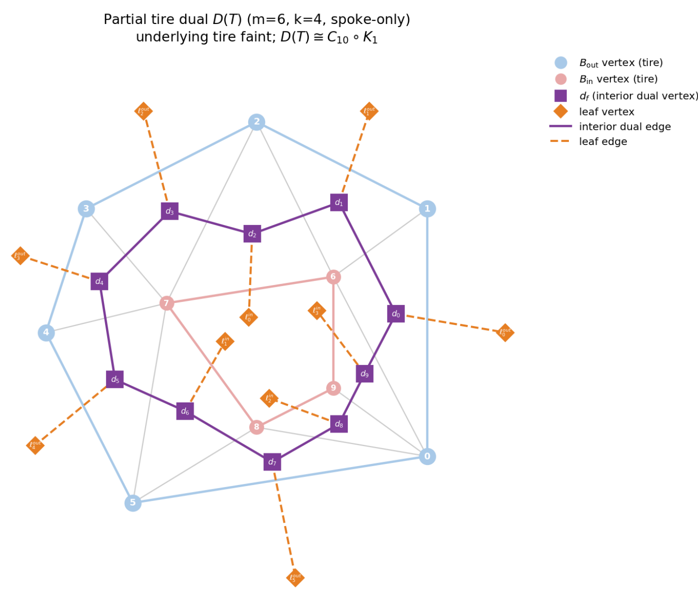

FIGURE 1. The partial tire dual $D(T)$ (purple squares + orange diamonds) drawn on top of a small tire graph T (faint) with $m = 6$ and $k = 4$. The ten interior vertices d_f at the centroids of the annular triangles form a single 10-cycle (solid purple); each boundary edge of the tire tread (either of B_{out} or of B_{in}) contributes a degree-1 leaf (orange diamond) attached to the unique annular face incident to it (dashed orange), giving the structure $C_{10} \circ K_1$ of Proposition 1.2.

Proposition 1.2 (Structure of $D(T)$ when the annular triangulation is spoke-only)**.** *Suppose B_{out} is a simple cycle of length n, O is a 2-connected outerplanar graph whose outer-face cycle B_{in} has length m, and E_{ann} consists only of* spokes *(edges*

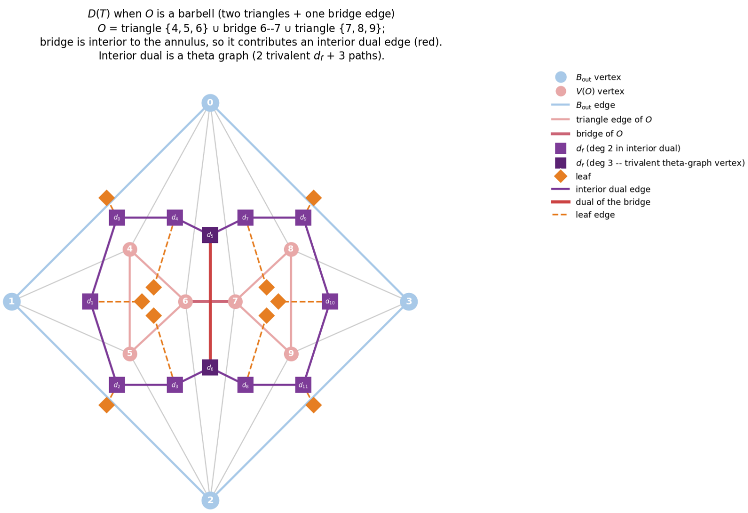

FIGURE 2. Partial tire dual $D(T)$ when the inner outerplanar graph O has a bridge — here a non-trivial edge cut connecting two disjoint triangles. B_{out} is a 4-cycle on $\{0, 1, 2, 3\}$ and O is the barbell: triangle $\{4, 5, 6\}$ together with triangle $\{7, 8, 9\}$ joined by the bridge edge 6–7 (removing the bridge disconnects O). Because both faces incident to the bridge are annular triangles, the bridge contributes an *interior dual edge* (highlighted in red) rather than two leaves; consequently the interior dual subgraph is no longer the single $(n+m)$-cycle of Proposition 1.2, but a theta graph: the two trivalent vertices d_5, d_6 (the bridge-incident annular faces) are joined by three internally vertex-disjoint paths in $D(T)$. Leaves come only from B_{out} ($n = 4$ leaves) and the six non-bridge edges of O ($m_\partial = 6$ leaves, three for each triangle).

with one endpoint in $V(B_{\mathrm{out}})$ and one in $V(B_{\mathrm{in}})$). Then each face $f \in F_{\mathrm{ann}}$ has exactly one boundary edge (on B_{out} or B_{in}) and two interior annular edges, and consequently $D(T)$ is isomorphic to the corona graph $C_{n+m} \circ K_1$: a cycle of length $n+m$ on the interior vertices $\{d_f\}$, with one leaf attached to each cycle vertex.

In particular, $|V(D(T))| = 2(n+m)$ and $|E(D(T))| = 2(n+m)$.

Proof. Each annular triangle f in a spoke-only triangulation has the form $\{x, y, z\}$ with $x \in V(B_{\mathrm{out}})$, $y \in V(B_{\mathrm{in}})$, and z also in $V(B_{\mathrm{out}}) \cup V(B_{\mathrm{in}})$. Of its three edges, the one between the two same-side vertices (x-z if both on B_{out}, or y-z if both on B_{in}) is a boundary edge of the tire tread; the other two edges are spokes.

So each d_f has degree 3 in $D(T)$: two from interior edges (= spokes shared with adjacent annular faces) and one leaf. The induced subgraph on $\{d_f : f \in F_{\mathrm{ann}}\}$ is 2-regular; together with the connectedness of the tire tread this forces it to be a single cycle. By [2, Remark 1.6], the cycle has length $n+m$, and there are also $n+m$ leaves attached one-per-cycle-vertex. □

Proposition 1.3 (Source-side simple-cycle property)**.** *Let G be a maximal planar graph with planar embedding Π_G and single-vertex source v_0. Let $d \geq 1$, $v \in L_d$, and let C' be a connected component of G'_d such that v is incident to some face in $F_{C'}$. Then the depth-d faces in $F_{C'}$ incident to v form a single contiguous arc in v's rotation in Π_G.*

Equivalently: for any such component, the source-side boundary of $R_{C'}$ is a simple cycle in L_d (no cut-vertices at level d).

Proof. Suppose for contradiction that the depth-d faces in $F_{C'}$ at v lie in two or more disjoint arcs of v's rotation. Adjacent vertices in G differ in level by at most 1, so a face at v has depth exactly d iff both other vertices have level $\geq d$, and depth $\leq d-1$ iff at least one has level $d-1$. Hence the gaps between the depth-d arcs at v are populated by level-$(d-1)$ neighbours of v, occurring in at least two disjoint arcs of v's rotation. Pick p in one such gap and q in another.

The BFS ball $G[L_{<d}]$ is connected, so there exists a simple path P in $G[L_{<d}]$ from p to q. Define the closed walk

$$W \;:=\; v \to p \to P \to q \to v.$$

Every vertex of P lies in $L_{<d}$, while $\ell(v) = d$, so v is distinct from every vertex of P; P is simple, so its internal vertices are distinct; and $p \neq q$ since they lie in different gaps. Hence W is a simple cycle in G.

By the Jordan curve theorem, the planar embedding of W divides Π_G into two regions. In v's rotation, the edges $v-p$ and $v-q$ lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by W. By choice of p, q, the two arcs of depth-d faces at v in $F_{C'}$ lie in different regions of W (i.e., one arc on each side).

Since C' is connected in G' and contains depth-d faces in both arcs, there is a dual path $f_1, f_2, \ldots, f_k$ in G'_d with $f_1, f_k \in F_{C'}$ incident to v in different arcs, and with the intermediate faces $f_2, \ldots, f_{k-1}$ not incident to v (a shortest such dual path). Consecutive faces f_i, f_{i+1} share an edge e_i of G; for $i \geq 2$, both endpoints of e_i lie in $L_{\geq d}$ (since neither f_i nor f_{i+1} is incident to v, all six vertices of these two triangles lie in $L_{\geq d}$). In particular, e_i shares no endpoint with W except possibly v — and v is excluded from $f_2, \ldots, f_{k-1}$.

A planar edge with neither endpoint on a simple closed planar curve W has both of its incident faces on the same side of W. Applying this to each e_i ($i \geq 2$) inductively: starting from f_2 on the same side of W as f_1 (their shared edge $e_1 = w - w'$ opposite to v in f_1 has $w, w' \in L_{\geq d}$ and hence is not on W), the path $f_2 \to f_3 \to \cdots \to f_{k-1} \to f_k$ stays on one side of W.

But f_1 and f_k lie on different sides of W (by construction), contradicting the conclusion that the entire path lies on one side. □

Lemma 1.4 (Tire-component lemma)**.** *Let G be a maximal planar graph and let $S \subseteq V(G)$ be a level source. Fix a plane embedding Π_G of G in which S lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For $d \geq 0$, let*

$$G'_d \;:=\; G'\bigl[\,\{d_f \in V(G') : \delta_G(d_f) = d\}\,\bigr]$$

be the inner-dual subgraph on dual vertices of dual depth d, and let C' be a connected component of G'_d. Write $F_{C'} := \{f : d_f \in V(C')\}$, $V_{C'} := \bigcup_{f \in F_{C'}} V(f)$, and let $C := G[V_{C'}]$ inherit its embedding from Π_G. Set $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$.

Then C, with the inherited embedding, is a tire graph in the sense of [2, Definition 1.5]*. Its outer boundary B_{out} is the side of $R_{C'}$ closer to S in Π_G, namely the level-d subgraph $G[V_{C'} \cap L_d]$ (a simple cycle or single vertex); its inner outerplanar graph is $O = G[V_{C'} \cap L_{d+1}]$, and its inner boundary B_{in} is the outer-face boundary closed walk of O in the inherited embedding (a simple cycle when O is 2-connected, a non-simple closed walk in general). The triangular faces of C inside the closed boundary region are exactly the faces of G in $F_{C'}$.*

Proof. Outerplanarity of the two level parts. By construction S lies on the outer face of Π_G, so the outerplanarity lemma of [1] applies directly with (G, Π_G, S), giving that $G[L_{d'}]$ is outerplanar for each $d' \geq 0$. Subgraphs of outerplanar graphs are outerplanar, so $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are both outerplanar.

Layer containment. Each $f \in F_{C'}$ has at least one vertex at level d, and adjacent vertices in G differ in level by at most 1; combined with $\delta_G(d_f) = d$, this forces $V(f) \subseteq L_d \cup L_{d+1}$. Hence $V_{C'} \subseteq L_d \cup L_{d+1}$, and C has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$.

Boundary edges are monochromatic in level. Each edge e on $\partial R_{C'}$ separates a face $f \in F_{C'}$ from a face $f' \notin F_{C'}$. Because f and f' share the edge e, their dual vertices are adjacent in G'; if both had depth d they would lie in the same component of G'_d, contradicting $d_f \in C'$ and $d_{f'} \notin C'$. Hence $\delta_G(d_{f'}) \neq d$; combined with the bounded-step property of δ across G'-adjacent faces, $\delta_G(d_{f'}) \in \{d-1, d+1\}$.

- If $\delta_G(d_{f'}) = d-1$, the third vertex w of $f' = \{u, v, w\}$ (where u, v are the endpoints of e) has $\ell(w) = d-1$. Each of u, v has $\ell \in \{d, d+1\}$ (from $V(f) \subseteq L_d \cup L_{d+1}$) and is adjacent to w, forcing $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$.
- If $\delta_G(d_{f'}) = d+1$, then all three vertices of f' lie in $L_{\geq d+1}$, so in particular $\ell(u) = \ell(v) = d+1$.

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of $\partial R_{C'}$ is monochromatic in level.

Boundary structure. Each connected component of $\partial R_{C'}$ traces a closed walk in G that, by the monochromaticity above, lies entirely in L_d or entirely in L_{d+1}. By Proposition 1.3, the depth-d faces of $F_{C'}$ at any $v \in L_d \cap V_{C'}$ form a single contiguous arc in v's rotation, so the source-side boundary walk visits each L_d-vertex of $V_{C'}$ exactly once: it is a simple cycle. At vertices $v \in L_{d+1} \cap V_{C'}$ the depth-d faces may split into multiple arcs of v's rotation; this corresponds exactly to v being a cut-vertex of O, and the inner-side boundary walk visits v correspondingly many times — which is already accommodated by [2, Definition 1.5] (where B_{in} is the outer-face boundary closed walk of O, not necessarily a simple cycle).

Outer boundary. Because S lies on the outer face of Π_G, the boundary curve(s) of $R_{C'}$ on the L_d side are closer to S in the embedding. In the inherited embedding of C, the unique unbounded face is the merged region containing the rest of Π_G outside $R_{C'}$ on the S side, so its boundary — a simple cycle on L_d (or a single vertex when $V_{C'} \cap L_d = \{v_0\}$, the $d = 0$ case) — serves as B_{out}. We set $B_{\mathrm{out}} := G[V_{C'} \cap L_d]$ if this is a cycle, and the single vertex $\{v_0\}$ in the degenerate case.

Inner outerplanar graph. By the outerplanarity lemma of [1], $G[V_{C'} \cap L_{d+1}]$ is outerplanar. We set $O := G[V_{C'} \cap L_{d+1}]$. The boundary curve(s) of $R_{C'}$ on the L_{d+1} side are exactly the boundary of O's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around O from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary B_{in}. No further restriction on O's internal structure is needed: when $R_{C'}$ has more than two boundary components in the surface-classification sense (i.e. several disjoint simple cycles on L_{d+1}), these correspond precisely to the multiple connected components or bridge crossings of O, and the outer-face boundary closed walk of O captures them collectively.

Tire structure. The triangular faces of C inside the closed boundary region are by construction the depth-d faces in $F_{C'}$, and the edges of C are $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$ where E_{ann} are the edges of G between $V_{C'} \cap L_d$ and $V_{C'} \cap L_{d+1}$ that bound a face of $F_{C'}$. □

Theorem 1.5 (Tire treads partition the bounded faces)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a level source lying on the outer face. For each $d \geq 0$ and each connected component C' of G'_d, let $T^{(d,C')}$ denote the tire graph supplied by Lemma 1.4, with tire tread $R_{C'} \subseteq |\Pi_G|$. Then the collection of treads*

$$\mathcal{R}(G,S) \;:=\; \big\{\, R_{C'} \,:\, d \geq 0,\ C' \text{ a connected component of } G'_d \,\big\}$$

partitions the bounded part of $|\Pi_G|$:

(i) *every bounded face f of G is contained in exactly one tread $R_{C'} \in \mathcal{R}(G,S)$;*
(ii) *distinct treads in $\mathcal{R}(G,S)$ have disjoint interiors and may share only boundary edges or vertices.*

Proof. Existence and uniqueness. Each bounded face $f \in F(G)$ has a uniquely-defined dual depth $\delta_G(d_f) \in \mathbb{Z}_{\geq 0}$, so the dual vertex d_f lies in G'_d for $d = \delta_G(d_f)$ and in no other $G'_{d'}$. Within G'_d, the vertex d_f belongs to exactly one connected component C'. By Lemma 1.4, $F_{C'}$ is precisely the set of faces $f' \in F(G)$ with $d_{f'} \in V(C')$; in particular $f \in F_{C'}$, hence $f \subseteq R_{C'}$.

For any other tread $R_{C''} \in \mathcal{R}(G,S)$, the component C'' is either at a different depth $d' \neq d$ (in which case $F_{C''}$ consists of depth-d' faces and $f \notin F_{C''}$) or at depth d but a different component $C'' \neq C'$ (in which case the two components are vertex-disjoint in G'_d, so again $f \notin F_{C''}$). In both cases $f \notin R_{C''}$ (more precisely, f is not one of the triangular faces of G in $F_{C''}$, so f's interior is not contained in $R_{C''}$).

Disjoint interiors. Each tread $R_{C'}$ is the union of its triangular faces $F_{C'} \subseteq F(G)$; distinct treads correspond to disjoint $F_{C'}$ (by the argument above), and the interiors of distinct G-faces are disjoint. Hence interiors of distinct treads are disjoint.

Coverage. Conversely, every bounded $f \in F(G)$ has $d_f \in V(G')$ with some dual depth d, and thus lies in $R_{C'}$ where C' is its component of G'_d. So $\bigcup_{R \in \mathcal{R}(G,S)} R$ contains every bounded face of G. □

Remark 1.6. Either boundary part of C in Lemma 1.4 may be degenerate. At $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ as the degenerate *outer* boundary and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G) as the inner boundary. Symmetrically, at $d = D_{\max}$, $V_{C'} \cap L_{D_{\max}+1} = \emptyset$ degenerates to a single deepest vertex serving as the *inner* boundary, with the level-$D_{\max}$ cycle as the outer boundary.

Remark 1.7. Two structural features of $R_{C'}$ that might at first appear to obstruct the tire-graph conclusion are both already accommodated by [2, Definition 1.5]:

Cut-vertices of O. A vertex $v \in V_{C'} \cap L_{d+1}$ may have the faces of $F_{C'}$ incident to it split into two or more arcs in v's rotation in Π_G, separated by faces of higher depth. This corresponds exactly to v being a cut-vertex of $O = G[V_{C'} \cap L_{d+1}]$, and the inner boundary closed walk B_{in} then visits v multiple times — once for each arc. No additional hypothesis is needed.

Multi-hole topology of $R_{C'}$. Even when $R_{C'}$ encloses several disjoint depth-$> d$ sub-regions, the inner outerplanar graph O captures the multi-hole structure as a disconnected or non-2-connected outerplanar graph (with bridges or multiple components), and its outer-face boundary closed walk serves as B_{in} traversing bridges twice and visiting cut-vertices multiple times.

In the special case $d = 0$ with single-vertex source $S = \{v_0\}$, $R_{C'}$ is the star of v_0, a topological closed disk with one boundary cycle (the link of v_0); the corresponding tire graph has degenerate outer boundary $\{v_0\}$.

Proposition 1.8 (Edge–vertex coloring bijection for $D(T)$)**.** *Let T be a tire graph satisfying the spoke-only hypothesis of Proposition 1.2 (so $D(T) \cong C_{n+m} \circ K_1$). Let $\Gamma \cong C_{n+m}$ be the interior dual subgraph of $D(T)$ induced on the interior dual vertices $\{d_f : f \in F_{\text{ann}}\}$. Then the number of proper 3-edge-colorings of $D(T)$ equals the number of proper 3-vertex-colorings of Γ, both given by*

$$2^{n+m} + 2 \cdot (-1)^{n+m}.$$

Proof. Write $L = n + m$, $\Gamma = C_L$. We construct mutually inverse bijections.

Step 1: proper 3-edge-colorings of $D(T) \leftrightarrow$ proper 3-edge-colorings of C_L. Given a proper 3-edge-coloring χ of $D(T)$, the three edges incident to any d_f carry three distinct colors; in particular the two cycle edges incident to d_f carry distinct colors, so $\chi|_{E(C_L)}$ is a proper 3-edge-coloring of C_L. Conversely, given a proper 3-edge-coloring ψ of C_L, the two cycle edges at any d_f have distinct colors, so a unique third color is available; assign that color to d_f's leaf edge. The resulting extension to $D(T)$ is proper at every d_f and vacuously proper at every leaf (degree 1), and the two maps are inverse to each other. Therefore

$$\#\{\text{proper 3-edge-colorings of } D(T)\} = \#\{\text{proper 3-edge-colorings of } C_L\}.$$

Step 2: proper 3-edge-colorings of $C_L \leftrightarrow$ proper 3-vertex-colorings of $L(C_L) \cong C_L$. The line graph $L(C_L)$ of a cycle of length L is again a cycle of length L; proper edge-colorings of C_L are by definition proper vertex-colorings of $L(C_L)$.

Step 3: count. The chromatic polynomial of the cycle is $P(C_L, k) = (k-1)^L + (-1)^L(k-1)$; at $k = 3$ this gives $2^L + 2 \cdot (-1)^L$. □

Remark 1.9. Proposition 1.8 reduces counting proper 3-edge-colorings of $D(T)$ to counting proper 3-vertex-colorings of a single cycle, giving a closed form $2^{n+m} + 2(-1)^{n+m}$ that depends only on $n + m$ (not on the specific spoke-only annular triangulation, nor on the chord structure of O). The count is preserved under the corona-with-K_1 structure of Proposition 1.2 precisely because each degree-1 leaf imposes no proper-edge-coloring constraint on itself; its color is freely determined as the missing third color at its attached interior vertex.

Definition 1.10 (Tire annular subgraph)**.** Let G be a maximal planar graph with embedding Π_G and inner planar dual G' (as in [2, Definition 1.3] above). Let

$T = (B_{\rm out}, O, E_{\rm ann}) \subseteq G$ be a tire graph ([2, Definition 1.5]), and let $F_{\rm ann} \subseteq F(G)$ denote its set of annular faces. The *tire annular subgraph* of T in G' is

$$T'_{\rm ann} \;:=\; G'\big[\,\{d_f : f \in F_{\rm ann}\}\,\big],$$

the subgraph of G' induced on the dual vertices corresponding to the annular faces of T. We equip $T'_{\rm ann}$ with the planar embedding inherited from G' (which, by deletion of vertices outside the annulus, remains a planar embedding of $T'_{\rm ann}$ in the sense of Π_G).

Definition 1.11 (Tire annular face connector)**.** With G, G', T as in Definition 1.10, let f' be a face of the tire annular subgraph $T'_{\rm ann}$ in its inherited embedding, and let $V(f') \subseteq V(T'_{\rm ann})$ denote the set of vertices on the boundary walk of f'. The *tire annular face connector at* f' is the subgraph

$$T'_{f'} \;:=\; \big(\,V(f') \cup N_{G'}(V(f'))\,,\;\{\,e \in E(G') : e \text{ is incident to } V(f')\,\}\,\big) \;\subseteq\; G',$$

i.e. the subgraph of G' on the closed G'-neighborhood of $V(f')$ together with every G'-edge incident to $V(f')$.

Definition 1.12 (Inner and outer spokes)**.** With $T'_{f'}$ as in Definition 1.11, regard f' as an open region of $|\Pi_G|$ and write $\overline{f'}$ for its closure. The vertices of $V(T'_{f'}) \setminus V(f')$ lie in $|\Pi_G| \setminus \overline{f'}$ or in f' (never on $\partial f'$, since the boundary walk of f' is by definition the set $V(f')$). Partition

$$V(T'_{f'}) \setminus V(f') \;=\; V_{\rm out}(T'_{f'}) \sqcup V_{\rm in}(T'_{f'})$$

where

$$\begin{aligned} V_{\rm out}(T'_{f'}) &:= \{\, v \in V(T'_{f'}) \setminus V(f') \;:\; v \notin \overline{f'}\,\},\\ V_{\rm in}(T'_{f'}) &:= \{\, v \in V(T'_{f'}) \setminus V(f') \;:\; v \in f'\,\}. \end{aligned}$$

The elements of $V_{\rm out}(T'_{f'})$ are the *outer spokes* of $T'_{f'}$ (vertices not in $V(f')$ that lie outside the region bounded by f'); the elements of $V_{\rm in}(T'_{f'})$ are the *inner spokes* of $T'_{f'}$ (vertices not in $V(f')$ that lie inside the region bounded by f').

Remark 1.13. In the spoke-only setting of Proposition 1.2, the tire annular subgraph is $T'_{\rm ann} = \Gamma \cong C_{n+m}$ (Proposition 1.8). This cycle has exactly two faces in its inherited embedding – one on each side of the cycle in Π_G – and both face boundaries traverse all $n+m$ vertices, so $V(f') = V(\Gamma)$ for either choice of f'. Each interior dual vertex d_f has G'-degree 3 (since G is a triangulation), of which two edges lie in Γ (cycle edges) and one edge points to a single non-annular face of G. Consequently $T'_{f'}$ has $n+m$ interior vertices plus the non-annular face vertices to which they connect, and is independent of the choice of f'. When G consists only of the tire T together with one source-side face inside $B_{\rm out}$ and one O-side face inside $B_{\rm in}$, $T'_{f'}$ recovers the planar dual of T itself.

2. A conjectural Latin-style substructure

Empirical enumeration (notes `tire_fiber_data.tex`, `tire_fiber_chords.tex`, `tire_fiber_step2.tex`, `tire_fiber_step2_large.tex`) of edge 3-coloring distributions on the tire annular face connector $T'_{f'}$ across 46 adjacent-tire pairs at $|\gamma| \in \{3, 4, 5, 6, 9, 12\}$ suggests that the chain-pigeonhole step on a shared cycle always succeeds. The data points to a structural mechanism: every edge-3-colourable

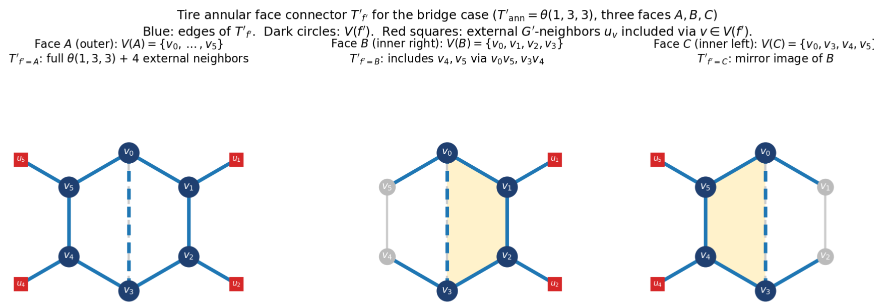

FIGURE 3. The bridge case: $T'_{\mathrm{ann}} = \theta(1,3,3)$ has three faces A, B, C in its inherited embedding, with respective vertex sets $V(A) = \{v_0, \ldots, v_5\}$, $V(B) = \{v_0, v_1, v_2, v_3\}$, and $V(C) = \{v_0, v_3, v_4, v_5\}$. In the surrounding maximal planar G, the chord endpoints v_0, v_3 (the two annular faces sharing the bridge edge) have all three G'-edges inside T'_{ann}, while each non-chord vertex v_i ($i \in \{1,2,4,5\}$) contributes one G'-edge to an external non-annular neighbor u_i. Each panel highlights $T'_{f'}$ (blue) inside G': dark circles are $V(f')$, gray circles are G'-neighbors of $V(f')$ within T'_{ann}, and red squares are external G'-neighbors u_i. The choice of face f' controls which external neighbors u_i are pulled into $T'_{f'}$ (face A pulls in all four; face B pulls in u_1, u_2 and face C pulls in u_4, u_5).

tire admits at least one "Latin-flavoured" boundary configuration, and adjacent tires share this same substructure on their common cycle.

Concretely, fix a tire T with inner outerplanar graph O on $V(B_{\mathrm{in}})$ and let $F(O)$ be the set of O-faces (in the tire's plane embedding, not counting the outer face B_{in}). For each O-face $f \in F(O)$, let $E_{\mathrm{in}}(f) \subseteq E(B_{\mathrm{in}})$ denote the set of B_{in} edges on f's boundary. In the Steiner-poor surrounding triangulation (where each O-face is a single face of G and dualises to a single G'-vertex of degree $|E_{\mathrm{in}}(f)|$ in $T'_{f'}$), proper edge 3-colouring of $T'_{f'}$ requires every O-face to have $|E_{\mathrm{in}}(f)| \leq 3$.

Let $\sigma_{B_{\mathrm{in}}}$ denote the spoke colouring restricted to the $|V(B_{\mathrm{in}})|$ inner-direction spoke positions on the dual annular cycle (equivalently: σ indexed by $E(B_{\mathrm{in}})$). Define the *Latin-flavoured set* on $\gamma = B_{\mathrm{in}}$ as

$$\mathcal{L}(B_{\mathrm{in}}, O) := \left\{ \sigma : E(B_{\mathrm{in}}) \to \{1,2,3\} \;\middle|\; \sigma\big|_{E_{\mathrm{in}}(f)} \text{ is a permutation of } \{1,2,3\} \text{ for every } f \in F(O) \right\}.$$

That is, on every O-face's B_{in}-edge boundary, all three colours appear exactly once (forcing $|E_{\mathrm{in}}(f)| = 3$ for each face — the maximally constrained case).

Conjecture 2.1 (Latin-substructure conjecture)**.** *For any Steiner-poor edge-3-colourable tire T with inner outerplanar graph O such that every O-face has exactly 3 B_{in}-edges, the realisable inner-spoke projection $\pi_D(\mathcal{C}(T'_{f'}))$ contains $\mathcal{L}(B_{\mathrm{in}}, O)$ as a subset. Moreover, $\mathcal{L}(B_{\mathrm{in}}, O)$ is invariant under the S_3 action on colours and has size at least $3! = 6$.*

Conjecture 2.2 (Chain-pigeonhole compatibility from Latin substructure)**.** *Adjacent tires T_1, T_2 sharing a cycle γ admit a joint edge 3-colouring whenever their respective inner-outerplanar structures $O^{(1)}, O^{(2)}$ both satisfy Conjecture 2.1. Equivalently: $\pi_D^{(1)}(\mathcal{C}(T_1)) \cap \pi_U^{(2)}(\mathcal{C}(T_2)) \supseteq \mathcal{L}(\gamma, O^{(1)}) \cap \mathcal{L}(\gamma, O^{(2)})$, and this last intersection is non-empty whenever the two face partitions of $E(\gamma)$ induced by $O^{(1)}, O^{(2)}$ share a common "Latin completion."*

The structural origin of these conjectures is the empirical observation that the smallest tested intersections on γ are always exactly the 3! permutations of a single canonical pattern in which each O-face on γ's side receives a permutation of $\{1, 2, 3\}$. For example, at $|\gamma| = 12$ with $O^{(1)}$ given by the chord matching $\{(0, 3), (4, 7), (8, 11)\}$ (face structure $\{0, 1, 2\} \sqcup \{4, 5, 6\} \sqcup \{8, 9, 10\} \sqcup \{3, 7, 11\}$), the canonical pattern $(1, 2, 3, 2, 2, 1, 3, 3, 2, 3, 1, 1)$ assigns the permutation $(1, 2, 3)$ to the first face, $(2, 1, 3)$ to the second, $(2, 3, 1)$ to the third, and $(2, 3, 1)$ to the fourth. Every face receives all three colours.

A proof of Conjecture 2.1 would convert the chain-pigeonhole compatibility step into a structural theorem on $T'_{f'}$: it is not the rough abundance of valid spoke configurations that lets adjacent tires meet, but a specific Latin-square-flavoured substructure dictated by the face partition of each tire's inner outerplanar graph. See `notes/tire_fiber_step2_large.tex` for the data underlying this conjecture and `experiments/tire_fiber_counterexample_search.log` for the ongoing automated search.

References

[1] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.
[2] E. Bauerfeld, *Coloring Nested Tire Graphs*, manuscript (math-research repository), 2026.