

# COLORED EDGE FLIP CLASSES

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ABSTRACT.

## 1. MOTIVATION

The Four Color Theorem asserts that every planar graph is properly 4-colorable, or equivalently that no maximal planar graph  $G$  satisfies  $\chi(G) \geq 5$ . Suppose, towards a contradiction, that such a graph exists; let  $G_0$  be one of minimum order. Any structural property shared by every maximal planar graph  $H$  with  $|V(H)| = |V(G_0)|$  is then automatically inherited by  $G_0$ , and any property *not* satisfied by  $G_0$  excludes a portion of the class of maximal planar graphs from playing the role of a minimum counterexample.

This paper investigates one such property: behavior under an edge flip. Our principal observation (Theorem 4.4) is that every edge flip of a minimum-order 5-chromatic maximal planar graph yields a 4-colorable graph. In particular, no such graph is *flip-symmetric*, where we call a maximal planar graph  $G$  flip-symmetric when some admissible flip at an edge of  $G$  returns a graph isomorphic to  $G$ . The search for a counterexample to the Four Color Theorem may therefore be confined to the complement of the class  $\mathcal{F}$  of flip-symmetric graphs.

## 2. PRELIMINARIES

Let  $G$  be a maximal planar graph with  $|V(G)| \geq 4$ , embedded in the plane so that every face — including the outer face — is a triangle. Every edge  $uv \in E(G)$  is then shared by exactly two triangular faces  $uvw$  and  $uvx$  whose union is a quadrilateral  $uwvx$  with diagonal  $uv$ .

**Definition 2.1** (Edge flip). Let  $G$  be a maximal planar graph and let  $uv \in E(G)$  be an edge whose two incident triangular faces are  $uvw$  and  $uvx$ . The *edge flip* (or *diagonal flip*) at  $uv$  is the operation that deletes the edge  $uv$  and inserts the edge  $wx$  in its place, replacing the two triangles  $uvw$  and  $uvx$  by the two triangles  $uwv$  and  $vwx$ . The flip is *admissible* if  $wx \notin E(G)$ ; otherwise the resulting multigraph is not simple and the flip is forbidden.

## 3. FLIP-SYMMETRIC MAXIMAL PLANAR GRAPHS

For a maximal planar graph  $G$  and an admissible edge  $uv \in E(G)$  with incident triangles  $uvw$ ,  $uvx$ , write

$$G^{\text{flip}(uv)} = (V(G), (E(G) \setminus \{uv\}) \cup \{wx\})$$

for the graph obtained from  $G$  by flipping  $uv$ .

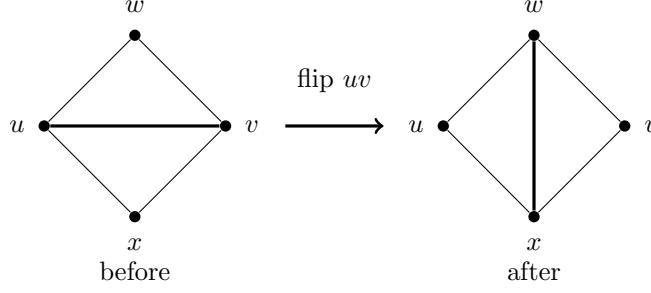


FIGURE 1. An edge flip replaces the diagonal  $uv$  of the quadrilateral  $uwvx$  with the diagonal  $wx$ .

**Definition 3.1** (Flip-symmetric graph). A maximal planar graph  $G$  is *flip-symmetric* if there exists an admissible edge  $uv \in E(G)$  such that  $G^{\text{flip}(uv)} \cong G$ . We write  $\mathcal{F}$  for the class of flip-symmetric maximal planar graphs.

**Definition 3.2** (Flip neighborhood). Let  $G$  be a maximal planar graph. The *flip neighborhood* of  $G$  is the set

$$\mathcal{N}(G) = \{ G^{\text{flip}(uv)} : uv \in E(G) \text{ and the flip at } uv \text{ is admissible} \}$$

of maximal planar graphs obtainable from  $G$  by a single admissible edge flip.

**Definition 3.3** (Colored edge flip class). Let  $G$  be a maximal planar graph and let  $\varphi$  be a proper 4-coloring of  $G$ . The *colored edge flip class* of  $(G, \varphi)$  is the set  $\mathcal{C}(G, \varphi)$  of maximal planar graphs reachable from  $G$  by some (possibly empty) sequence of admissible edge flips, each of which leaves  $\varphi$  a proper 4-coloring of the resulting graph. Explicitly,  $H \in \mathcal{C}(G, \varphi)$  iff there exist graphs  $G = G_0, G_1, \dots, G_k = H$  such that for each  $0 \leq i < k$ ,  $G_{i+1} = G_i^{\text{flip}(u_i v_i)}$  for some  $u_i v_i \in E(G_i)$  whose flip is admissible in  $G_i$  and whose opposite vertices  $w_i, x_i$  satisfy  $\varphi(w_i) \neq \varphi(x_i)$ .

#### 4. A MINIMAL FOUR-COLORABLE COUNTEREXAMPLE

**Definition 4.1** (Edge-deletion subgraph). Let  $G$  be a maximal planar graph and  $uv \in E(G)$ . The *edge-deletion subgraph at  $uv$*  is the spanning subgraph  $G - uv = (V(G), E(G) \setminus \{uv\})$ . Write  $\mathcal{D}(G) = \{G - uv : uv \in E(G)\}$ .

**Lemma 4.2.** *Let  $G_0$  be a maximal planar graph of minimum order with  $\chi(G_0) \geq 5$ . Then every  $H \in \mathcal{D}(G_0)$  is 4-colorable.*

*Proof.* Fix  $uv \in E(G_0)$  and let  $G_0/uv$  denote the simple planar graph obtained by contracting  $uv$  and discarding parallel edges. Then  $G_0/uv$  is a simple planar graph on  $|V(G_0)| - 1 \geq 4$  vertices but is not in general a triangulation; triangulate its planar embedding (by adding chords inside any non-triangular face) to obtain a maximal planar graph  $T$  on the same vertex set, with  $G_0/uv$  as a spanning subgraph and  $|V(T)| < |V(G_0)|$ . By the minimality of  $G_0$ ,  $T$  admits a proper 4-coloring, which restricts to a proper 4-coloring  $c$  of  $G_0/uv$ . Let  $z$  be the contracted vertex and define  $c' : V(G_0) \rightarrow \{1, 2, 3, 4\}$  by  $c'(u) = c'(v) = c(z)$  and  $c'(y) = c(y)$  for  $y \notin \{u, v\}$ . Every edge of  $G_0 - uv$  is either disjoint from  $\{u, v\}$  or incident to exactly one of them; in either case the corresponding edge of  $G_0/uv$  has distinct

endpoints under  $c$ , so  $c'$  assigns its endpoints distinct colors. The edge  $uv$  itself is absent from  $G_0 - uv$ , so  $c'$  is a proper 4-coloring of  $G_0 - uv$ .  $\square$

**Lemma 4.3.** *Let  $G_0$  be a maximal planar graph of minimum order with  $\chi(G_0) \geq 5$ , fix  $uv \in E(G_0)$ , and let  $\varphi$  be any proper 4-coloring of  $G_0 - uv$ . Write  $a = \varphi(u)$  and let  $b, c, d$  denote the three remaining colors. Then:*

- (1)  $\varphi(v) = a$ ;
- (2) the subgraph of  $G_0 - uv$  induced by the vertices of color  $a$  or  $b$  contains a path from  $u$  to  $v$ ;
- (3) the subgraph of  $G_0 - uv$  induced by the vertices of color  $a$  or  $c$  contains a path from  $u$  to  $v$ .

*Proof.* (1) If  $\varphi(v) \neq a$  then  $\varphi$  is already a proper 4-coloring of  $G_0$ , since the only edge of  $G_0$  absent from  $G_0 - uv$  is  $uv$  and its endpoints have distinct colors. This contradicts  $\chi(G_0) \geq 5$ , so  $\varphi(v) = a$ .

(2) Suppose, for contradiction, that  $u$  and  $v$  lie in distinct connected components of the subgraph of  $G_0 - uv$  induced by the color classes  $a$  and  $b$ . Let  $C$  be the component containing  $u$ , and define  $\varphi': V(G_0) \rightarrow \{a, b, c, d\}$  by swapping colors  $a \leftrightarrow b$  on  $C$  and leaving every other vertex unchanged. Then  $\varphi'$  is a proper 4-coloring of  $G_0 - uv$  with  $\varphi'(u) = b$  and  $\varphi'(v) = a$ , contradicting part (1) applied to  $\varphi'$ .

(3) Identical to (2) with  $c$  in place of  $b$ .  $\square$

**Theorem 4.4.** *Let  $G$  be a minimum-order maximal planar graph with  $\chi(G) \geq 5$ . Then every  $H \in \mathcal{N}(G)$  is 4-colorable.*

*Proof.* Fix an edge  $e = uv \in E(G)$ , and let  $F_0, F_1$  be the two triangular faces of  $G$  incident to  $e$ , so that  $\{w, x\} = (V(F_0) \cup V(F_1)) \setminus \{u, v\}$ . By Lemma 4.2,  $G - e$  admits a proper 4-coloring  $\varphi$ .

*Case 1:*  $\varphi(w) \neq \varphi(x)$ . Then  $\varphi$  is also a proper 4-coloring of the graph induced by the edge flip of  $e$ .

*Case 2:*  $\varphi(w) = \varphi(x)$ . Set  $a = \varphi(u)$ ; by Lemma 4.3(1),  $\varphi(v) = a$  as well, and the edges  $uw, vw \in E(G - e)$  force  $\varphi(w) \neq a$ . Choose a color  $b \notin \{a, \varphi(w)\}$ . By Lemma 4.3, there is a path  $P$  from  $u$  to  $v$  in the subgraph of  $G - e$  induced by the vertices of color  $a$  or  $b$ . Let  $\{c, d\} = \{1, 2, 3, 4\} \setminus \{a, b\}$ ; then  $\varphi(w) = \varphi(x) \in \{c, d\}$ .

Any path from  $w$  to  $x$  in the subgraph of  $G - e$  induced by the vertices of color  $c$  or  $d$  would, in the plane embedding of  $G - e$ , cross  $P$ ; but its vertices have colors in  $\{c, d\}$  and the vertices of  $P$  have colors in  $\{a, b\}$ , and these sets are disjoint, so the two paths share no vertex. Hence  $w$  and  $x$  lie in distinct connected components of the  $\{c, d\}$ -colored subgraph of  $G - e$ . Swapping colors  $c \leftrightarrow d$  on the component containing  $w$  yields a proper 4-coloring of  $G - e$  in which  $\varphi(w) \neq \varphi(x)$ , reducing to Case 1.  $\square$

**Theorem 4.5.** *Let  $G$  be a minimum-order maximal planar graph with  $\chi(G) \geq 5$ . Then for every  $H \in \mathcal{N}(G)$  and every proper 4-coloring  $\varphi$  of  $H$ ,*

$$G \notin \mathcal{C}(H, \varphi).$$

*Proof.* Suppose, for contradiction, that  $G \in \mathcal{C}(H, \varphi)$  for some  $H \in \mathcal{N}(G)$  and some proper 4-coloring  $\varphi$  of  $H$ . By Definition 3.3, there exists a sequence of maximal planar graphs  $H = H_0, H_1, \dots, H_k = G$  in which each  $H_{i+1}$  is obtained from  $H_i$

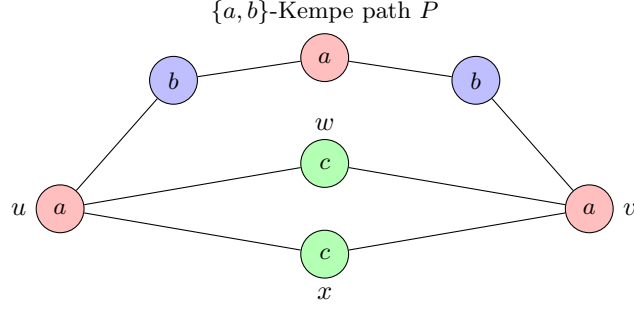


FIGURE 2. Case 2 of the proof of Theorem 4.4:  $u, v$  share color  $a$  and  $w, x$  share color  $c$ . The  $\{a, b\}$ -Kempe path  $P$  from  $u$  to  $v$  separates  $w$  from  $x$  in the plane, so no  $\{c, d\}$ -path between  $w$  and  $x$  can avoid crossing  $P$ ; since the color sets  $\{a, b\}$  and  $\{c, d\}$  are disjoint, no such path exists.

by an admissible edge flip that leaves  $\varphi$  a proper 4-coloring of  $H_{i+1}$ . By induction on  $i$ ,  $\varphi$  is a proper 4-coloring of every  $H_i$ ; in particular,  $\varphi$  is a proper 4-coloring of  $H_k = G$ . But  $\chi(G) \geq 5$  admits no such coloring, a contradiction.  $\square$