Cut-and-depth-label: a procedure for labelling half-graphs of a 6-edge cut by “distance to the cut”

Procedure

Given a maximal planar graph G and its dual G':

1. Find a 6-edge cut $C \subseteq E(G')$ partitioning $V(G')$ into S and $V \setminus S$ with both sides non-empty.
2. Remove the cut edges to obtain two graphs $G'_0 := G'[S] \cup \emptyset_C$ and $G'_1 := G'[V \setminus S] \cup \emptyset_C$ (each a cubic graph minus boundary edges).
3. In each G'_i:
   (a) Let V_i be the set of vertices of degree 2 in G'_i (i.e. original cubic vertices incident to exactly one cut edge; the cut edge accounts for the missing edge, so the vertex sits at degree $3 - 1 = 2$).
   (b) For each $v \in V_i$, attach a new pendant edge to a fresh vertex, and **label these pendant edges with depth** 0.
   (c) For $d = 0, 1, 2, \ldots$: label every unlabelled edge that shares a vertex with a depth-d edge with depth $d + 1$.
   (d) Stop when every edge has a depth label.

Interpretation. The depth labels give a BFS distance in the line graph of G'_i starting from the pendants added at the cut. Equivalently, the depth of an edge e in G'_i is the minimum number of edges traversed (via shared-vertex adjacency) to reach a pendant.
Caveat. If the cut C is not a *matching* cut (i.e., the 6 cut edges share vertices), then some boundary vertices have degree < 2 in G'_i and do not receive a pendant under the strict reading of step 3(a). When the cut is a matching, each of the 12 boundary vertices (6 per side) has degree exactly 2 and receives a pendant; the construction is symmetric.

Example: Holton-McKay graph #0

We apply the procedure to the first of the six non-Hamiltonian 38-vertex cubic plane graphs found by Holton and McKay (loaded from `papers/even_level_graph_generators/experiments/nonham38m4.pc`). This G' is itself a cubic plane graph; its dual G is a 21-vertex triangulation.

The chosen 6-edge cut. Greedy search over 128 distinct 6-edge cuts in G', preferring *matching cuts* (both sides have 6 distinct boundary vertices) and then balanced $|S|$, returns

$$|S| = 10, \qquad C = \{(34, 29), (35, 30), (26, 22), (27, 23), (28, 24), (31, 25)\}.$$

This is a matching cut: the 6 edges have 12 distinct endpoints, 6 on each side.

Resulting half-graphs.

	G'_0	G'_1
$\lvert S\rvert$	10	28
Original vertices in G'_i	10	28
$\lvert V_i\rvert$ (pendants added)	6	6
Total vertices in G'_i	16	34
Total edges	18	45
Max depth assigned	2	7

Multi-cut vertices (degree < 2 in induced subgraph): *none* on either side, since the cut is a matching.

Visualization.

Cut tires

The depth labelling on G'_i organises its edges into layers indexed by distance to the cut. Each layer gives rise to a family of "cut tires" that play the same structural role as the tires of `paper.tex` but are derived from the cut rather than from a level source in the primal G.

Definition (Cut tire)**.** *Let G'_i be a cut half (Step 3 above) with edge depth labelling* depth : $E(G'_i) \to \mathbb{Z}_{\geq 0}$. *Fix $d > 0$ and let $H_d \subseteq G'_i$ be the subgraph induced on the edges of depth d (vertex set = endpoints of depth-d edges, edges = depth-d edges). Equip H_d with the planar embedding inherited from G'_i.*

For each face f of H_d, the cut tire at (d, f) *is the subgraph of G'_i consisting of:*

- *every edge on the boundary walk of f (all of depth d in H_d), and*
- *every edge of G'_i with at least one endpoint on the boundary walk of f whose depth is $d-1$ or $d+1$.*

The first set forms the face boundary at depth *d; the second splits into* inner spokes *(depth $d-1$, pointing toward the cut) and* outer spokes *(depth $d+1$, pointing away from the cut).*

Relation to the existing tire framework. Under the correspondence between primal level structure (paper.tex) and dual depth labelling, a cut tire on G'_i at (d, f) corresponds to:

- face boundary at depth $d \longleftrightarrow$ the tire annular subgraph T'_{ann} at depth d from the cut (cf. Def. 1.15 of paper.tex);
- the cut tire itself $\longleftrightarrow$ the tire annular face connector $T'_{f'}$ (cf. Def. 1.16);
- inner / outer spokes $\longleftrightarrow$ the inner and outer spokes of $T'_{f'}$ (cf. Def. 1.17).

So the cut tire is the dual-side analogue of the "tire annular face connector," parametrised by depth from the cut rather than depth from a primal level source.

Example on G'_1 (Holton-McKay #0, $V \setminus S$ half). In this example:

- $d = 1$: face length 12, 5 inner spokes (to depth-0 pendants) + 4 outer spokes. This is the outermost cut tire, immediately adjacent to the cut.
- $d = 2$: face length 7 (one of two symmetric faces in H_2), $4 + 3$ spokes.
- $d = 4$: face length 8, $2 + 5$ spokes.
- $d = 5$: face length 14, $4 + 6$ spokes.
- $d = 6$: face length 12, $7 + 1$ spokes. This is the innermost cut tire (one face left, almost no outer spokes).

Connection to chain pigeonhole / 4CT reducibility

The procedure mirrors the 4CT cut-and-reglue scheme (`rainbow_proof.tex`, `worst_case_proof_sketch.tex`, `two_approaches_comparison.tex`) at the structural level. After cutting, each G'_i is a cubic-minus-boundary graph; the pendant additions formally restore cubicity at the 6 degree-2 boundary vertices. The depth label on each edge measures its "distance to the cut."

For a minimum counterexample to the 4CT (i.e., a cubic plane G' with no proper 3-edge-colouring), the depth labels organise each G'_i into concentric layers indexed by distance to the cut. The 3-edge-colourings of G'_i must extend a colouring at the depth-0 pendants (= a ring colouring at the cut); the BFS ordering by depth is the natural induction order for propagating the colouring inward.

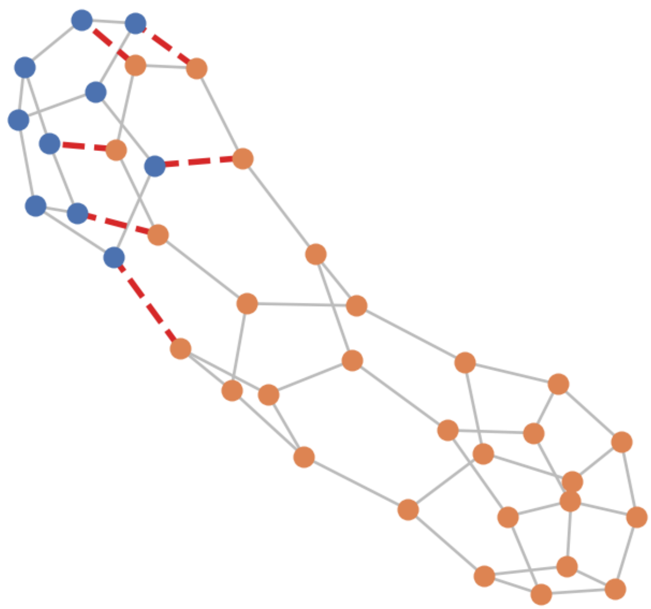
In the tire framework, the cut cycle γ in the primal G (corresponding to the 6-edge cut in G' via planar duality) plays the role of the tire's inner boundary on one side and outer boundary on the other. The depth label on G'_i-edges is exactly the dual analogue of plane depth from γ (cf. the level-cycle generalization discussion in the recent conversation).

Limitations of this example

- Holton-McKay graphs are cyclically 5-edge-connected (not 6-edge-connected), so 6-edge cuts are not the minimum cyclic cut. The smallest cyclic edge cuts in this graph are size 5.
- The matching 6-cut found is highly imbalanced ($|S| = 10$ vs. $|S^c| = 28$). Searching among the 128 distinct 6-edge cuts for a balanced matching cut may give better examples.
- Depth labels propagate via the line graph, not via vertex BFS. An alternative procedure would label *vertices* by BFS distance from the boundary; both yield similar layered structures but with slightly different counts.

Cut-and-depth-label procedure on Holton-McKay graph #0
Same vertex positions used across all three panels.

G' = Holton-McKay #0 with 6-edge cut highlighted
Blue = S ($|S| = 10$); orange = $V \setminus S$ ($|V \setminus S| = 28$); red dashed = cut



G'_0 (|S| = 10, |V| = 6, max depth = 2)

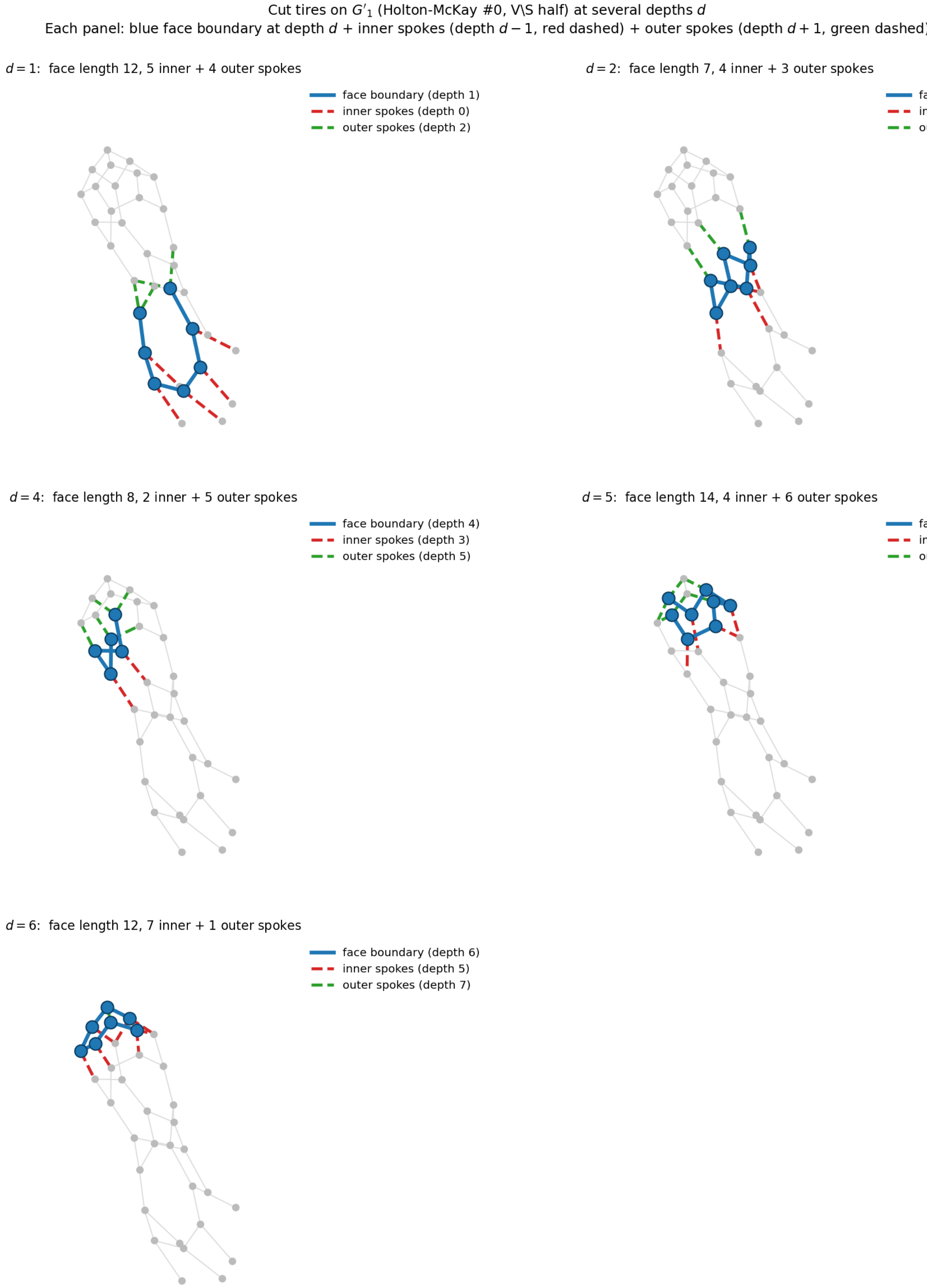


Figure 2: Cut tires on G_1' at depths $d = 1, 2, 4, 5, 6$. In each panel, blue solid edges form the face boundary at depth d; red dashed edges are inner spokes (depth $d-1$); green dashed edges are outer spokes (depth $d+1$). Vertices on the face boundary are highlighted; the rest of G_1' is faded.