Counterexamples to the loose chain pigeonhole conjecture (and what they suggest about the right statement)

The conjecture as stated

From `cut_depth_label.tex`, the loose chain pigeonhole conjecture says:

> For every non-trivial cut tire T (with ≥ 1 in/out spoke), the joint projection $\pi(T)$ is non-empty, S_3-closed, and contains at least one full S_3-orbit (i.e. $|\pi(T)| \geq 6$).

Counterexamples already in the data

Re-examining the cut tire tests on G'_1 of Holton-McKay #0 (`cut_tire_conjecture_tests.tex`):

d	face	out	in	total spokes	$\lvert\pi(T)\rvert$	orbit sizes
1	0	5	0	5	93	$[3, 6^{15}]$
1	1	1	0	1	3	$[3]$
2	0	4	3	7	126	$[6^{21}]$
2	1	4	3	7	126	$[6^{21}]$
4	0	1	0	1	3	$[3]$
4	1	2	1	3	21	$[3, 6^3]$
6	0	3	2	5	93	$[3, 6^{15}]$

Two non-trivial cut tires have $|\pi(T)| = 3 < 6$: $(d, \text{face}) = (1, 1)$ and $(4, 0)$. Both have exactly 1 spoke. In each case, $\pi(T)$ is the 3-element S_3-orbit of a single-color spoke configuration — which has S_3-stabilizer of order 2 (the two permutations fixing the chosen color and swapping the others), so the orbit has size $|S_3|/2 = 3$.

Why this happens (and is general)

For a cut tire with exactly k in/out spokes, the projection $\pi(T) \subseteq \{1, 2, 3\}^k$. S_3 acts by permuting colors. The S_3-orbit of $\sigma \in \{1, 2, 3\}^k$ has size:

- 3 if σ uses exactly 1 color (stabilizer of order 2);
- 6 if σ uses ≥ 2 distinct colors (trivial stabilizer).

For $k = 1$: every $\sigma \in \{1, 2, 3\}$ uses exactly 1 color, so every orbit has size 3, and $|\pi(T)| \in \{0, 3\}$. The conjecture's "$|\pi(T)| \geq 6$" is automatically false for any non-empty $\pi(T)$ on a 1-spoke tire.

For $k \geq 2$, σ can use 1 or ≥ 2 colors:

- Single-color σ contributes a size-3 orbit.

- Multi-color σ contributes size-6 orbits.

$|\pi(T)| \geq 6$ requires at least one multi-color σ, which is not automatic but typically holds when k is large or the tire allows enough flexibility.

Why the conjecture's chain half can still fail

Even if we restrict to $k \geq 2$ cut tires (avoiding the trivial $k = 1$ counterexample), the chain pigeonhole conclusion is not immediate:

- Chain composition involves *compatibility* between adjacent cut tires. A full S_3-orbit at one layer projects through the in/out-spoke face-boundary bijection to some subset at the adjacent layer; this projection might land in a size-3 orbit even if the source was size-6.
- The chain at $d_{\max}$ terminates (no in spokes); the innermost cut tire's structure constrains everything outward.
- Sufficiency for $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$ requires the chain to preserve the S_3-orbit structure all the way to the cut, on both sides, and have a common orbit there.

The counterexamples at 1-spoke tires don't break the chain at those depths because they're at non-essential "side" faces (face 1 at $d = 1$, face 0 at $d = 4$; the main chain runs through the larger faces). But they show the per-tire claim is not universal.

Refined conjectures

In view of these counterexamples, three natural refinements:

1. **Restrict to ≥ 2 spokes.** "For every cut tire T with ≥ 2 in/out spokes total, $|\pi(T)| \geq 6$ and $\pi(T)$ contains a full S_3-orbit." This avoids the $k = 1$ trivial case. Still empirically open whether $k = 2$ cut tires always have ≥ 6 projections.
2. **Weaken to non-emptiness.** "For every non-trivial cut tire, $\pi(T)$ is non-empty and S_3-closed." This is much weaker but might still suffice for chain pigeonhole *provided* chain composition preserves non-emptiness — which is itself the hard step.
3. **Allow size-3 orbits.** "For every non-trivial cut tire, $\pi(T)$ is non-empty and S_3-closed; the orbits in $\pi(T)$ have size 3 or 6." This is just a description of the empirical findings, not a useful conjecture — chain pigeonhole at a size-3-orbit layer is more restrictive than at a size-6 one.

Where to look for harder counterexamples

The two counterexamples found are at low-spoke tires. More substantive counterexamples would look like:

- A $k \geq 2$ spoke cut tire whose face boundary is structured such that only single-color σ's extend, leaving $|\pi(T)| = 3$. This would require the face boundary to force all spoke colors equal — an unusual constraint.

- A chain in which adjacent compatibility kills all size-6 orbits, leaving only size-3 at the cut layer. This would be a chain-level counterexample to the bottom-line pigeonhole.

Neither is observed in the present data. But the data is from a single 6-cut on a single Holton-McKay graph. A broader empirical sweep (multiple cuts, multiple Holton-McKay graphs, larger G') might surface harder counterexamples.

Suggested next step. Restrict the conjecture to $k \geq 2$ tires and re-run the empirical test. If still empirically true after broader sweeps, attempt to prove the restricted form via the existing Prop 1.13 lower bound (which gives $\geq 2^n + 2(-1)^n$ colorings for spoke-only cut tires of cycle length n) plus a combinatorial argument that the multi-color σ's are non-empty.