

# FACE-MONOCROMATIC PAIRS AND THE FOUR COLOUR THEOREM

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**ABSTRACT.** We propose the *face-monochromatic-pair conjecture*, a structural property of proper 3-edge-colourings of cubic plane graphs that, if true, implies the Four Colour Theorem. Working in the planar dual  $G'$  of a hypothetical minimal counterexample  $G$  to 4CT, we delete a single pentagonal face of  $G'$  and rewire its five external vertices around a new apex vertex and a chord; the resulting *reduced dual*  $\widehat{G}'_{v,i}$  is a smaller cubic plane graph whose proper 3-edge-colourings, by the minimality of  $G$ , are constrained by a chord-apex condition and a pair of Kempe-cycle conditions. The face-monochromatic-pair conjecture, in its strengthened form, asserts the existence in every such colouring of a face  $F$  and two non-incident same-coloured edges  $e_1, e_2 \in \partial F$  whose subdivision-and-bridging produces a 4-face  $f_n$  whose boundary colouring places it under the hypothesis of a 4-face edge-suppression theorem; we use this theorem to derive a proper 3-edge-colouring of  $G'$ , contradicting minimality. We verify the conjecture computationally on all chord-apex+Kempe colourings of reduced duals with  $|V(G)| \leq 20$  (142,812 colourings, all pass); the weaker form is verified up to  $|V(G)| \leq 21$  (535,182 colourings, all pass).

## 1. INTRODUCTION

The Four Colour Theorem (4CT) — that every loopless plane graph admits a proper 4-vertex-colouring — has, since the late 1970s, only been proved by computer-assisted case analysis on a discharging argument over a finite set of unavoidable reducible configurations. Appel and Haken’s original proof [2, 3], the Robertson–Sanders–Seymour–Thomas reworking [4], and Gonthier’s machine-checked version [5] all share that structure.

This paper takes a different approach: rather than discharge over configurations in the triangulation  $G$ , we work in its planar dual  $G'$ , a cubic plane graph whose proper 3-edge-colourings correspond by Tait’s theorem to proper 4-vertex-colourings of  $G$ . Assuming  $G$  is a minimal counterexample to 4CT, we delete a single pentagonal face of  $G'$  and rewire its five external vertices, obtaining a smaller cubic plane graph  $\widehat{G}'_{v,i}$  — the *reduced dual* — which by minimality *is* properly 3-edge-colourable. Two structural lemmas constrain every such colouring: a *chord-apex* condition (Lemma 3.7) forcing two named edges to share a colour, and a pair of *Kempe-cycle* conditions (Lemma 3.8) placing four of the rewired edges on common bichromatic Kempe cycles. These constraints are the starting point of the present development.

The main contribution of the paper is the *face-monochromatic-pair conjecture* (Conjecture 5.1) and its strengthening (Conjecture 5.24), which we show together

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imply the Four Colour Theorem. The supporting ingredients are the chord-apex and Kempe-cycle lemmas on reduced-dual colourings, the classical operation of *edge suppression* (delete the edge and smooth its two degree-2 endpoints; equivalently, simple-graph contraction in the dual triangulation; recalled in Section 4), and an observation that suppression preserves 3-edge-colourability when applied across a 4-face whose two opposite boundary edges carry different colours (Theorem 4.2).

The strategy is to construct, from a putative minimal counterexample’s reduced-dual colouring, a 4-face  $f_n$  in a slightly modified graph  $\widehat{G}^{'+}$  to which the suppression theorem applies; the suppression then produces a properly 3-edge-coloured graph from which a 3-edge-colouring of  $G'$  can be recovered, contradicting the non-4-colourability of  $G$ . The face-monochromatic-pair conjecture asserts the existence of the structural data  $(F, e_1, e_2)$  needed to build  $f_n$ ; the strengthening guarantees that  $f_n$ ’s boundary colouring falls under the suppression theorem’s hypothesis. Both conjectures have been verified computationally on all chord-apex+Kempe colourings of reduced duals up to  $|V(G)| \leq 20$ , with the weaker form going up to  $|V(G)| \leq 21$ .

Organization. Section 2 fixes the minimal-counterexample framework:  $G$  is a triangulation,  $\delta(G) \geq 5$ , and every triangulation on fewer vertices is properly 4-colourable. Section 3 introduces the reduced dual  $\widehat{G}'_{v,i}$  and proves the chord-apex and Kempe-cycle lemmas. Section 4 defines edge suppression and proves its 4-face 3-edge-colourability theorem. Section 5 states the two conjectures, reports the empirical verification, and gives the implication to 4CT.

Companion paper. An iterated version of the reduced-dual construction — producing a sequence  $H_1, H_2, \dots$  of progressively smaller cubic plane graphs and tracking an accumulating “protected” edge set — is the subject of a companion paper. The present paper does not use that iteration.

## 2. THE MINIMAL COUNTEREXAMPLE

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

**Lemma 2.1** (Reduction to triangulations). *If every triangulation is properly 4-vertex-colourable, then so is every plane graph.*

*Proof.* Let  $H$  be a plane graph. Add edges to  $H$ , maintaining planarity, until no further edge can be added; the result is a triangulation  $H^+$  on the same vertex set with  $E(H) \subseteq E(H^+)$ . A proper 4-colouring of  $H^+$  restricts to a proper 4-colouring of  $H$ , since every edge of  $H$  is an edge of  $H^+$ .  $\square$

By Lemma 2.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

**Definition 2.2** (Minimal counterexample). Let  $G$  be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call  $G$  a *minimal counterexample*. By minimality, every triangulation on fewer than  $|V(G)|$  vertices is properly 4-colourable.

*Remark 2.3.* Since every triangulation on at most four vertices is properly 4-colourable (the largest being  $K_4$ ), a minimal counterexample has  $|V(G)| \geq 5$ ; the degree bound below sharpens this to  $|V(G)| \geq 12$ .

**Lemma 2.4** (Minimum degree). *A minimal counterexample  $G$  has minimum degree  $\delta(G) \geq 5$ .*

*Proof.* Suppose some vertex  $v$  has  $\deg(v) = d \leq 4$ .

If  $d \leq 3$ , let  $G' = G - v$ . Then  $G'$  is a plane graph on fewer vertices, so by Definition 2.2 and Lemma 2.1 it has a proper 4-colouring. The at most three neighbours of  $v$  use at most three colours, so a fourth colour is free for  $v$ , extending the colouring to  $G$  — a contradiction.

If  $d = 4$ , again 4-colour  $G - v$ . If the four neighbours of  $v$  use at most three colours we extend as before, so assume they receive all four colours; let  $v_1, v_2, v_3, v_4$  be the neighbours in cyclic order around  $v$ , coloured 1, 2, 3, 4. Consider the subgraph induced by the colour classes 1 and 3, and let  $K$  be its connected component containing  $v_1$ . If  $v_3 \notin K$ , swap colours 1 and 3 on  $K$ ; now no neighbour of  $v$  is coloured 1, freeing it for  $v$ . If  $v_3 \in K$ , then a 1–3 Kempe chain joins  $v_1$  to  $v_3$ , and this chain together with  $v$  encloses exactly one of  $v_2, v_4$ ; hence the 2–4 component containing  $v_2$  cannot also reach  $v_4$ , and swapping colours 2 and 4 on it frees colour 2 for  $v$ . Either way the colouring extends to  $G$ , a contradiction.

Hence  $\delta(G) \geq 5$ . □

### 3. THE REDUCED DUAL

Write  $G'$  for the dual of  $G$ : since  $G$  is a triangulation,  $G'$  is a cubic plane graph in which each vertex of  $G$  corresponds to a face of  $G'$ , each face of  $G$  to a vertex of  $G'$ , and each edge to a dual edge. A vertex of  $G$  of degree  $k$  corresponds to a  $k$ -gonal face of  $G'$ .

The following labelling of vertices in a properly 3-edge-coloured cubic plane graph will be useful in Section 5.

**Definition 3.1** (Heawood number of a vertex). Let  $H$  be a cubic plane graph with a fixed planar embedding, and let  $\varphi: E(H) \rightarrow \{1, 2, 3\}$  be a proper 3-edge-colouring. At each vertex  $v \in V(H)$ , the three incident edges receive three distinct colours; reading them in clockwise order around  $v$  gives a cyclic permutation of  $(1, 2, 3)$ . The *Heawood number* of  $v$  is

$$h_\varphi(v) := \begin{cases} +1 & \text{if the clockwise cyclic colour order at } v \text{ is } (1, 2, 3), \\ -1 & \text{if it is } (1, 3, 2). \end{cases}$$

Equivalently,  $h_\varphi(v) = +1$  when the clockwise colour order at  $v$  is an even cyclic permutation of  $(1, 2, 3)$  and  $-1$  when it is an odd one. The labels are due to Heawood [1], who introduced them as part of his analysis of 3-edge-colourings of cubic plane graphs.

By Lemma 2.4,  $\delta(G) \geq 5$ , and Euler's formula gives  $\sum_{u \in V(G)} (6 - \deg u) = 12$ , so  $G$  has a vertex of degree exactly 5 (indeed at least twelve). Fix such a vertex  $v$ . Its dual face  $F_v$  is a pentagon, bounded by the five dual vertices corresponding to the five faces of  $G$  incident to  $v$ .

**Definition 3.2** (Reduced dual). Let  $v$  be a degree-5 vertex of  $G$  with pentagonal dual face  $F_v$ , and fix an index  $i \in \{0, 1, 2, 3, 4\}$ . The *reduced dual*  $\widehat{G}'_{v,i}$  is the plane graph obtained from  $G'$  as follows.

- (1) Delete the five dual vertices on the boundary of  $F_v$ , together with all edges incident to them. Each deleted vertex is cubic, with two edges on  $\partial F_v$  and one edge leaving  $F_v$ ; deleting the five boundary vertices therefore removes the five external edges as well, dropping their five outer endpoints from degree 3 to degree 2. These five degree-2 vertices lie on the boundary of a single face  $F$  of the resulting graph.
- (2) List the five degree-2 vertices in clockwise order around  $F$  as  $A = (A_0, A_1, A_2, A_3, A_4)$ .
- (3) Add a new vertex  $v_n$  and join it to  $A_i, A_{i+1}$ , and  $A_{i+2}$  (indices mod 5) by three new edges.
- (4) Add a new edge between  $A_{i+3}$  and  $A_{i+4}$  (indices mod 5).

*Remark 3.3.* Steps (3) and (4) restore cubicity:  $A_i, A_{i+1}, A_{i+2}$  each gain one edge to  $v_n$  and  $A_{i+3}, A_{i+4}$  each gain the new edge, so all five return to degree 3, and  $v_n$  has degree 3. Since  $A_i, \dots, A_{i+2}$  and  $A_{i+3}, A_{i+4}$  are each consecutive along  $\partial F$ , the new vertex and edge can be drawn inside  $F$  without crossings, so  $\widehat{G}'_{v,i}$  is again a cubic plane graph. The construction depends on the choice of  $i$  up to the rotational symmetry of  $A$ .

**Definition 3.4** (Edges of the reduced dual). The four edges added in steps (3) and (4) of Definition 3.2 are named as follows. The chord  $A_{i+3}A_{i+4}$  is the *merged edge*; the edge  $A_{i+1}v_n$  is the *spike edge*; the edge  $A_i v_n$  is the *side-0 edge*; and the edge  $A_{i+2}v_n$  is the *side-1 edge*. In the  $i = 0$  case of Figure 1 these are  $\{A_3, A_4\}$ ,  $\{A_1, v_n\}$ ,  $\{A_0, v_n\}$ , and  $\{A_2, v_n\}$  respectively.

We will use the following structural fact about proper 3-edge-colourings near a pentagonal face of a cubic plane graph; it is stated for a generic such graph  $H$ , not specifically for the reduced dual.

**Lemma 3.5** (Pentagonal externals). *Let  $H$  be a cubic plane graph and  $F$  a pentagonal face of  $H$ , with  $\partial F$  traversed clockwise as  $u_0, u_1, u_2, u_3, u_4$ . For each  $i$  let  $f_i$  be the unique edge of  $H$  incident to  $u_i$  that does not lie on  $\partial F$ . An assignment  $\varphi$  of colours from  $\{1, 2, 3\}$  to the ten edges incident to  $\{u_0, \dots, u_4\}$  is proper at every  $u_i$  if and only if there is some index  $j$  such that*

$$\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2}) \quad \text{and} \quad \{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1, 2, 3\} \setminus \{\varphi(f_j)\},$$

*indices mod 5.*

*Proof.* Write  $e_i = u_i u_{i+1}$  for the boundary edges of  $\partial F$  (indices mod 5). A colouring  $\varphi$  is proper at every  $u_i$  if and only if at each  $u_i$  the three incident edges  $e_{i-1}, e_i, f_i$  receive three distinct colours; whenever this holds,  $\varphi(f_i)$  is forced to be the unique colour in  $\{1, 2, 3\} \setminus \{\varphi(e_{i-1}), \varphi(e_i)\}$ , and  $\varphi$  restricts to a proper 3-edge-colouring of the cycle  $\partial F$ .

( $\Rightarrow$ ) The line graph of  $\partial F$  is  $C_5$ , whose maximum independent set has size 2, so no colour appears more than twice on  $\partial F$ ; and since  $\partial F$  is an odd cycle, all three colours appear. The colour multiset on  $(\varphi(e_0), \dots, \varphi(e_4))$  is therefore  $(2, 2, 1)$ , with the singleton at a unique position. Cyclically shifting indices we may place this position at 0; let  $c$  be the singleton colour. The remaining four edges form the path



FIGURE 1. The four steps of Definition 3.2, illustrated on  $G' =$  the dodecahedron (dual of the icosahedron) with  $F_v$  the inner pentagon and  $i = 0$ . Top left: delete the five boundary vertices of  $F_v$ , leaving five degree-2 vertices on a new face  $F$ . Top right: order them clockwise as  $A_0, \dots, A_4$ . Bottom left: add  $v_n$  joined to  $A_0, A_1, A_2$ . Bottom right: add the chord  $A_3A_4$ , giving the cubic plane graph  $\hat{G}'_{v,0}$ .

$e_1e_2e_3e_4$ , which by propriety alternates between the other two colours, so for some labelling  $\{a, b, c\} = \{1, 2, 3\}$ ,

$$(\varphi(e_0), \varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)) = (c, a, b, a, b).$$

Reading off the forced values of  $\varphi(f_i)$ ,

$$\varphi(f_0) = a, \quad \varphi(f_1) = b, \quad \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c,$$

which is the lemma's pattern at  $j = 2$  (the cyclic shift maps this back to the corresponding  $j$  in the original indexing). This case is the unique proper 3-edge-colouring of  $\partial F$  up to cyclic shift and permutation of  $\{1, 2, 3\}$  (since  $5 \cdot 3! = 30 = P(C_5, 3)$ , the chromatic polynomial of  $C_5$  at 3), so it exhausts every proper  $\varphi$ .

( $\Leftarrow$ ) The lemma's hypothesis is invariant under cyclic shifts of indices and under permutations of  $\{1, 2, 3\}$ , so we may assume  $j = 2$ ,  $\varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c$ ,  $\varphi(f_0) = a$ , and  $\varphi(f_1) = b$ , with  $\{a, b, c\} = \{1, 2, 3\}$ . Propriety at  $u_i$  and  $u_{i+1}$  requires  $\varphi(e_i) \notin \{\varphi(f_i), \varphi(f_{i+1})\}$ , which gives

$$\varphi(e_0) = c, \quad \varphi(e_1) = a, \quad \varphi(e_2) \in \{a, b\}, \quad \varphi(e_3) \in \{a, b\}, \quad \varphi(e_4) = b.$$

The remaining propriety condition  $\varphi(e_{i-1}) \neq \varphi(e_i)$  holds automatically at  $u_0, u_1, u_4$ , forces  $\varphi(e_2) = b$  at  $u_2$ , and then forces  $\varphi(e_3) = a$  at  $u_3$ . The resulting triples  $(\varphi(e_{i-1}), \varphi(e_i), \varphi(f_i))$  at  $u_0, u_1, u_2, u_3, u_4$  are

$$(b, c, a), \quad (c, a, b), \quad (a, b, c), \quad (b, a, c), \quad (a, b, c),$$

each a permutation of  $\{1, 2, 3\}$ , so  $\varphi$  is proper at every  $u_i$ .  $\square$

*Remark 3.6.* The two-element condition  $\{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1, 2, 3\} \setminus \{\varphi(f_j)\}$  cannot be dropped: a 3-colouring satisfying  $\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2})$  alone need not extend, e.g.  $(1, 1, 1, 1, 2)$ .

Since  $\widehat{G}'_{v,i}$  is the dual of a triangulation on fewer vertices than  $G$ , it is 3-edge-colourable by the minimality of  $G$ . The following lemma constrains every such colouring.

**Lemma 3.7.** *Let  $G$  be a minimal counterexample, and let  $\widehat{G}'_{v,i}$  be a reduced dual of its dual  $G'$ . Then in every proper 3-edge-colouring of  $\widehat{G}'_{v,i}$ , the merged edge and the spike edge receive the same colour.*

*Proof.* After cyclically relabelling, assume  $i = 0$ . Suppose for contradiction that  $\varphi$  is a proper 3-edge-colouring of  $\widehat{G}'_{v,0}$  in which the merged edge  $\{A_3, A_4\}$  and the spike edge  $\{A_1, v_n\}$  receive different colours (Figure 2, top), and write

$$X = \varphi(\{A_0, v_n\}), \quad Y = \varphi(\{A_1, v_n\}), \quad Z = \varphi(\{A_2, v_n\}), \quad W = \varphi(\{A_3, A_4\}).$$

Propriety of  $\varphi$  at  $v_n$  forces  $\{X, Y, Z\} = \{1, 2, 3\}$ , and the assumption  $W \neq Y$  leaves  $W \in \{X, Z\}$ .

We lift  $\varphi$  to a colouring  $\psi$  of  $E(G')$  as follows. Let  $B_0, \dots, B_4$  be the boundary vertices of  $\partial F_v$  in  $G'$ , indexed so that  $f_k = B_k A_k$ . On every edge of  $G'$  that survived the reduction, set  $\psi = \varphi$ . At each  $A_k$  the two surviving edges retain their  $\varphi$ -colours, so the remaining edge at  $A_k$  — in  $G'$  this is the external  $f_k$ ; in  $\widehat{G}'_{v,0}$  this is a  $v_n$ -edge ( $k \in \{0, 1, 2\}$ ) or the chord ( $k \in \{3, 4\}$ ) — is forced to take the third colour at  $A_k$ . Since the two-surviving-edge colours at  $A_k$  agree in  $G'$  and  $\widehat{G}'_{v,0}$ , the third colour does too, giving

$$\psi(f_0) = X, \quad \psi(f_1) = Y, \quad \psi(f_2) = Z, \quad \psi(f_3) = \psi(f_4) = W$$

(the last two equalities holding because the chord is a single edge contributing its colour at each of  $A_3$  and  $A_4$ ).

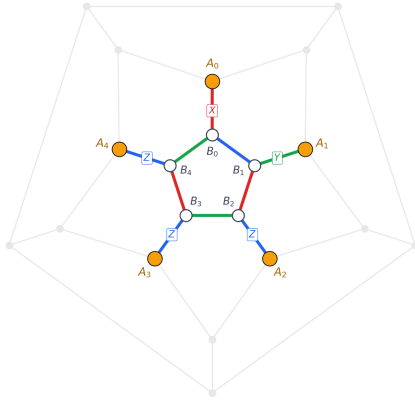
It remains to assign colours to the five boundary edges  $B_k B_{k+1}$  of  $\partial F_v$ . Apply Lemma 3.5 to  $G'$  at the face  $F_v$  with the  $B_k$ 's as its boundary vertices and the same indexing. The external vector  $(\psi(f_0), \dots, \psi(f_4)) = (X, Y, Z, W, W)$  falls into one of two cases (Figure 2, bottom):

- if  $W = Z$ , it is  $(X, Y, Z, Z, Z)$ : three consecutive  $Z$ 's at positions 2, 3, 4, with  $\{X, Y\} = \{1, 2, 3\} \setminus \{Z\}$ ;

Step 1:  $\phi$  on  $\widehat{G}_{v,0}$  assigns distinct colours  $X, Y, Z$  to the  $v_n$ -edges (propriety at  $v_n$ );  
by hypothesis  $W \neq Y$ , forcing  $W \in \{X, Z\}$ .



Step 2: lift to  $G'$  when  $W = Z$ . The externals inherit  $\psi(f) = (X, Y, Z, Z, Z)$ ;  
Lemma 2.4 colours the five edges of  $\partial F_v$ .



Step 3: lift to  $G'$  when  $W = X$ . The externals inherit  $\psi(f) = (X, Y, Z, X, X)$ ;  
Lemma 2.4 colours the five edges of  $\partial F_v$ .

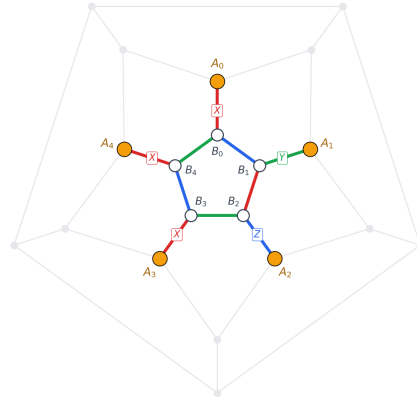


FIGURE 2. The proof of Lemma 3.7, illustrated for  $i = 0$  on  $G' =$  the dodecahedron. Top: under the assumption  $W \neq Y$ , propriety at  $v_n$  forces  $W \in \{X, Z\}$ . Bottom: in either case the lift to  $G'$  has externals satisfying the hypothesis of Lemma 3.5, which colours  $\partial F_v$  to extend  $\psi$  to a proper 3-edge-colouring of  $G'$ .

- if  $W = X$ , it is  $(X, Y, Z, X, X)$ : three consecutive  $X$ 's at positions 3, 4, 0, with  $\{Y, Z\} = \{1, 2, 3\} \setminus \{X\}$ .

Each case satisfies the hypothesis of Lemma 3.5; its  $(\Leftarrow)$  direction therefore assigns colours to the boundary edges  $B_k B_{k+1}$  that make  $\psi$  proper at every  $B_k$ .

The resulting  $\psi$  is a proper 3-edge-colouring of  $G'$ : proper at every  $B_k$  by the lemma, at every  $A_k$  by the construction, and at every other vertex because such a vertex has the same neighbourhood in  $G'$  as in  $\widehat{G}'_{v,0}$  with the same incident-edge colours. By Tait's theorem,  $G'$  is 3-edge-colourable iff  $G$  is 4-vertex-colourable, contradicting that  $G$  is a counterexample. The assumption  $W \neq Y$  is therefore false.  $\square$

For a pair of colours  $\{a, b\} \subseteq \{1, 2, 3\}$ , the subgraph of  $\widehat{G}'_{v,i}$  on the edges coloured  $a$  or  $b$  is 2-regular (since at each vertex exactly one of the three incident edges is excluded), and hence a disjoint union of cycles. We call each such cycle a  $\{a, b\}$ -Kempe cycle, and reserve the notation for the specific cycle containing a given edge when the context makes it clear. Swapping the two colours on a single Kempe cycle yields another proper 3-edge-colouring of the same graph.

**Lemma 3.8** (Kempe cycles through the spike). *Let  $G$  be a minimal counterexample, fix a reduced dual  $\widehat{G}'_{v,i}$  of  $G'$ , and let  $\varphi$  be a proper 3-edge-colouring of  $\widehat{G}'_{v,i}$ . Write  $c$  for the common colour assigned by  $\varphi$  to the spike and the merged edge (Lemma 3.7), and  $c_0, c_1$  for the colours of the side-0 and side-1 edges respectively, so  $\{c, c_0, c_1\} = \{1, 2, 3\}$ . Then*

- (1) *the  $\{c, c_0\}$ -Kempe cycle through the spike edge contains both the side-0 edge and the merged edge;*
- (2) *the  $\{c, c_1\}$ -Kempe cycle through the spike edge contains both the side-1 edge and the merged edge.*

*Proof.* We prove (1); (2) is the same argument with  $c_1$  and the side-1 edge in place of  $c_0$  and the side-0 edge.

The spike edge  $\{A_{i+1}, v_n\}$  and the side-0 edge  $\{A_i, v_n\}$  share the vertex  $v_n$  and receive the two colours  $c, c_0$ , so they both lie on the  $\{c, c_0\}$ -Kempe cycle through  $v_n$ . Suppose for contradiction that the merged edge lies on a different  $\{c, c_0\}$ -Kempe cycle  $K$  (it lies on *some* such cycle, since it has colour  $c$ ). Let  $\varphi'$  be obtained from  $\varphi$  by swapping the colours  $c$  and  $c_0$  along  $K$  alone: this is a Kempe swap, so  $\varphi'$  is again a proper 3-edge-colouring of  $\widehat{G}'_{v,i}$ . Under  $\varphi'$  the spike edge — which is not on  $K$  — still has colour  $c$ , but the merged edge — which is on  $K$  — now has colour  $c_0$ . Hence in  $\varphi'$  the spike and the merged edge receive distinct colours, contradicting Lemma 3.7 applied to  $\varphi'$ .  $\square$

#### 4. EDGE SUPPRESSION

We recall the classical operation of *edge suppression* on cubic plane graphs: delete the edge and smooth the two resulting degree-2 endpoints. Under planar duality this coincides with simple-graph contraction on the dual side. It will be the central tool in Section 5 below, where we formulate a sufficient condition for the Four Colour Theorem.

**Definition 4.1** (Edge suppression). Let  $H$  be a cubic plane graph and  $e = uv$  an edge of  $H$  with  $u \neq v$  and no edge of  $H$  parallel to  $e$ . The *edge suppression* of  $H$  at  $e$  is the graph  $H'$  obtained in two steps:

- (1) *Delete the edge  $e$ ; the endpoints  $u$  and  $v$  each drop to degree 2.*
- (2) *Smooth each of  $u$  and  $v$ : at  $u$ , replace  $u$  and its two remaining incident edges  $ua, ub$  by a single new edge  $ab$ ; do the same at  $v$ . Both vertices  $u$  and  $v$  are removed, and two new edges are added in their place.*



Provided the smoothings do not introduce a loop or parallel edge,  $H'$  is again a cubic plane graph, with  $|V(H')| = |V(H)| - 2$  and  $|E(H')| = |E(H)| - 3$ .

Equivalently,  $H'$  is the planar dual of  $\text{dual}(H)/e^*$ , where  $e^*$  is the edge of  $\text{dual}(H)$  crossing  $e$  and the contraction on the right-hand side is simple-graph contraction (loops removed, parallel edges absorbed). Under planar duality, contracting  $e^*$  in  $\text{dual}(H)$  merges the two triangular faces of  $\text{dual}(H)$  incident to  $e^*$ , and the parallel-edge cleanup corresponds exactly to the smoothing step on the primal side.



FIGURE 3. Edge suppression (Definition 4.1). Left: a fragment of a cubic plane graph with the suppressed edge  $e = uv$  highlighted in red. Middle: deleting  $e$  leaves  $u$  and  $v$  of degree 2. Right: smoothing  $u$  and  $v$  replaces each pair of incident edges by a single new edge, removing  $u, v$  and giving a cubic plane graph again.

**Theorem 4.2** (Edge suppression across a 4-face preserves 3-edge-colourability). *Let  $H$  be a cubic plane graph with a proper 3-edge-colouring  $\varphi$ , let  $f$  be a face of  $H$  with  $|\partial f| = 4$ , and let  $e_0, e_1$  be the two edges of  $\partial f$  sharing no endpoint (the opposite pair on the 4-cycle  $\partial f$ ). If  $\varphi(e_0) \neq \varphi(e_1)$  and the edge suppression of  $H$  at  $e_0$  (Definition 4.1) is well-defined (no loops or parallel edges are created), then the suppressed graph admits a proper 3-edge-colouring.*

*Proof.* Write  $\partial f$  as the 4-cycle  $v_0v_1v_2v_3$  with  $e_0 = v_0v_1$  and  $e_1 = v_2v_3$  (so  $e_0, e_1$  are opposite); the remaining two boundary edges of  $f$  are  $e_2 := v_1v_2$  and  $e_3 := v_3v_0$ . Since  $H$  is cubic, each  $v_i$  has exactly one edge not on  $\partial f$ : write  $w_i$  for that edge and  $u_i$  for its other endpoint, so  $w_i = v_iu_i$  with  $u_i \notin \{v_0, v_1, v_2, v_3\}$ , for each  $i \in \{0, 1, 2, 3\}$ . Put  $a := \varphi(e_0)$ ,  $b := \varphi(e_1)$ , and let  $c$  be the third colour.

*Forced colours on the face.* Propriety at  $v_1$  and  $v_2$  forces  $\varphi(e_2) \notin \{a, b\}$ , so  $\varphi(e_2) = c$ ; then  $\varphi(w_1) = b$  and  $\varphi(w_2) = a$ . Symmetrically  $\varphi(e_3) = c$ ,  $\varphi(w_0) = b$ , and  $\varphi(w_3) = a$ . In particular  $\varphi(w_0) = \varphi(w_1) = b$ .

*Construction of  $\varphi'$ .* Let  $H'$  denote the edge suppression of  $H$  at  $e_0$ ; its new edges are  $e'_3 := v_3u_0$  (replacing  $e_3$  and  $w_0$  via the smoothing at  $v_0$ ) and  $e'_2 := v_2u_1$  (replacing  $e_2$  and  $w_1$  via the smoothing at  $v_1$ ). Define  $\varphi': E(H') \rightarrow \{1, 2, 3\}$  by

$$\varphi'(e) := \begin{cases} c & \text{if } e = e_1, \\ b & \text{if } e \in \{e'_2, e'_3\}, \\ \varphi(e) & \text{otherwise.} \end{cases}$$

That is: give each smoothed-in edge the colour  $b$  (the colour of the two  $w_i$  it absorbs), recolour  $e_1$  to  $c$ , and leave every other edge of  $H'$  with its  $\varphi$ -colour.

*Propriety.* Every vertex of  $H'$  other than  $v_2, v_3, u_0, u_1$  has the same incident edges and the same  $\varphi'$ -colours as it did under  $\varphi$ , so propriety is inherited there. At

the four affected vertices,

vertex	edges in $H'$	colours under $\varphi'$
$v_2$	$e_1, w_2, e'_2$	$c, a, b$
$v_3$	$e_1, w_3, e'_3$	$c, a, b$
$u_0$	$e'_3, \alpha_0, \beta_0$	$b, a, c$
$u_1$	$e'_2, \alpha_1, \beta_1$	$b, a, c$

where  $\alpha_i, \beta_i$  are the two edges of  $H$  at  $u_i$  other than  $w_i$ , whose  $\varphi$ -colours are forced to  $\{a, c\}$  by propriety at  $u_i$  (since  $\varphi(w_i) = b$ ). Each row lists three distinct colours, so  $\varphi'$  is proper.  $\square$

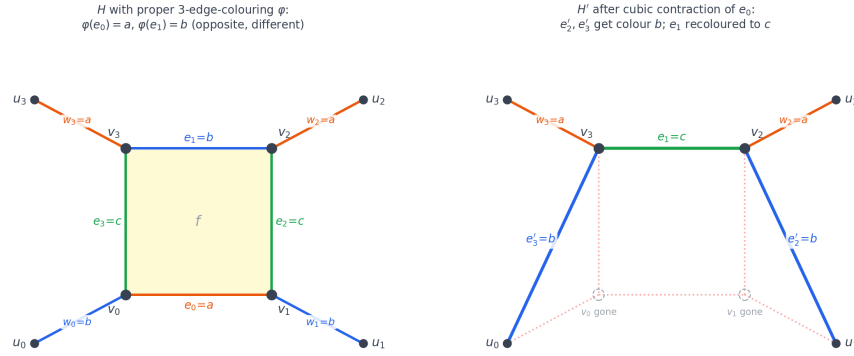


FIGURE 4. The recolouring used in the proof of Theorem 4.2. Left: the 4-face  $f$  of  $H$  under  $\varphi$ , with the forced colours  $\varphi(e_0) = a$ ,  $\varphi(e_1) = b$ ,  $\varphi(e_2) = \varphi(e_3) = c$ ,  $\varphi(w_0) = \varphi(w_1) = b$ , and  $\varphi(w_2) = \varphi(w_3) = a$ . Right: the suppressed graph  $H'$  under  $\varphi'$ . The smoothed-in edges  $e'_2, e'_3$  inherit the colour  $b$  from  $w_0, w_1$ , and  $e_1$  is recoloured from  $b$  to  $c$ ; every edge outside the face neighbourhood keeps its  $\varphi$ -colour (dotted in red: the five edges of  $H$  removed by the suppression).

## 5. THE FACE-MONOCROMATIC-PAIR CONJECTURE AND THE FOUR COLOUR THEOREM

The following conjecture identifies a structural property of every proper 3-edge-colouring of a reduced dual of a minimal counterexample. If true, it implies the Four Colour Theorem via Theorem 4.2.

**Conjecture 5.1** (Face-monochromatic-pair conjecture). *Let  $G$  be a minimal counterexample to the Four Colour Theorem, and let  $\hat{G}'_{v,i}$  be a reduced dual of  $G' = \text{dual}(G)$ . Then for every proper 3-edge-colouring  $\varphi$  of  $\hat{G}'_{v,i}$  there exist a face  $F$  of  $\hat{G}'_{v,i}$  and two distinct edges  $e_1, e_2 \in \partial F$ , with neither  $e_1$  nor  $e_2$  equal to the merged edge, such that:*

- (1)  $\varphi(e_1) = \varphi(e_2)$ . Write  $a := \varphi(e_1) = \varphi(e_2)$ .

- (2)  $e_1, e_2$ , and the merged edge all lie on a common  $\{a, b\}$ -Kempe cycle of  $\varphi$ , for some colour  $b \neq a$ .
- (3) Exactly one edge of  $\partial F$  lies between  $e_1$  and  $e_2$  along one of the two arcs of  $\partial F$ . Equivalently, subdividing  $e_1, e_2$  by new vertices  $X_1, X_2$  and joining them by a new edge  $X_1 X_2$  inside  $F$  creates a new face  $f_n$  bounded by exactly 4 edges (the new edge  $X_1 X_2$ , the two subdivision halves adjacent to it, and the single  $\partial F$ -edge between  $e_1$  and  $e_2$ ).

**Lemma 5.2** (A Heawood-constant Kempe cycle does not admit the clause-(3) arc). *Let  $G$  be a minimal counterexample to the Four Colour Theorem, fix a reduced dual  $\widehat{G}'_{v,i}$  of  $G' = \text{dual}(G)$ , and let  $\varphi$  be a proper 3-edge-colouring of  $\widehat{G}'_{v,i}$ . Set  $a := \varphi(\text{merged})$  and let  $K$  be the  $\{a, b\}$ -Kempe cycle of  $\varphi$  through the merged edge for some  $b \in \{1, 2, 3\} \setminus \{a\}$ . If  $h_\varphi$  is constant on  $V(K)$ , then no edge  $e \in E(K)$  admits a face  $F$  of  $\widehat{G}'_{v,i}$  and two non-incident edges  $e_1, e_2 \in \partial F$  such that  $\varphi(e_1) = \varphi(e_2)$  and  $e$  is the unique edge of  $\partial F$  between  $e_1$  and  $e_2$  along one of the two arcs of  $\partial F$  — that is, no edge of  $K$  admits the clause-(3) arc of Conjecture 5.1 with  $e_1, e_2$  at its two endpoints.*

*Proof.* Let  $c$  be the third colour. Fix any edge  $e \in E(K)$  joining  $v_0, v_1 \in V(K)$ . By hypothesis  $h_\varphi(v_0) = h_\varphi(v_1)$ ; after possibly relabelling we may take  $h_\varphi(v_0) = h_\varphi(v_1) = +1$ , so by Definition 3.1 the clockwise cyclic colour order at  $v_0$  and at  $v_1$  is the same even cyclic class  $(a, b, c)$ .

Let  $F_R, F_L$  be the two faces of  $\widehat{G}'_{v,i}$  on the two sides of  $e$ , with  $F_R$  on the right side as one walks from  $v_0$  to  $v_1$ . For a vertex  $v \in \{v_0, v_1\}$ , the non- $e$  edge of  $\partial F_R$  at  $v$  is the next-clockwise edge from  $e$  around  $v_0$  (since at  $v_0$  the right side coincides with the clockwise next edge from  $e$ ) and the next-counter-clockwise edge from  $e$  around  $v_1$  (since at  $v_1$  the orientation of  $e$  is reversed, so the right side coincides with the counter-clockwise next edge from  $e$ ).

*Case A:*  $\varphi(e) = a$ . In the CW order  $(a, b, c)$  at  $v_0$  the next-CW edge from  $e$  has colour  $b$ ; in the same CW order  $(a, b, c)$  at  $v_1$  the next-CCW edge from  $e$  has colour  $c$  (since CCW-next from  $a$  in cyclic order  $(a, b, c)$  is  $c$ ). Hence the non- $e$  edge of  $\partial F_R$  at  $v_0$  has colour  $b$ , while the non- $e$  edge of  $\partial F_R$  at  $v_1$  has colour  $c$  — these differ. Symmetrically, the non- $e$  edges of  $\partial F_L$  at  $v_0$  and  $v_1$  have colours  $c$  and  $b$  respectively, again different. Hence the colour- $b$  edges at  $v_0$  and  $v_1$  lie on opposite faces of  $e$ , and the same for the colour- $c$  edges; no face of  $\widehat{G}'_{v,i}$  contains two same-coloured non- $e$  edges at  $\{v_0, v_1\}$ .

*Case B:*  $\varphi(e) = b$ . By the analogous reasoning, the non- $e$  edges of  $\partial F_R$  at  $v_0$  and  $v_1$  have colours  $c$  and  $a$  respectively, and those of  $\partial F_L$  have colours  $a$  and  $c$ . The colour- $a$  edges at  $v_0, v_1$  thus lie on opposite faces of  $e$ , and so do the colour- $c$  edges.

In either case, no face  $F$  of  $\widehat{G}'_{v,i}$  has two same-coloured non- $e$  edges at  $\{v_0, v_1\}$  on  $\partial F$ , so the clause-(3) arc (with  $e$  as the unique  $\partial F$ -edge between  $e_1$  and  $e_2$  at the endpoints of  $e$ ) cannot be realised.  $\square$

**Lemma 5.3** (If Conjecture 5.1 fails, both Kempe cycles through merged have constant Heawood number). *Let  $G, \widehat{G}'_{v,i}, \varphi$  be as in Lemma 5.2, set  $a := \varphi(\text{merged})$ , and let  $K_b, K_c$  be the two Kempe cycles of  $\varphi$  through the merged edge — the  $\{a, b\}$ -Kempe cycle and the  $\{a, c\}$ -Kempe cycle, where  $\{b, c\} = \{1, 2, 3\} \setminus \{a\}$ . If no triple*



FIGURE 5. The two cases in the proof of Lemma 5.2. Vertices  $v_0, v_1$  are consecutive on the  $\{a, b\}$ -Kempe cycle  $K$ , joined by an edge  $e$ , with the lemma's hypothesis  $h_\varphi(v_0) = h_\varphi(v_1) = +1$  — so both vertices share the clockwise colour order  $(a, b, c)$ . *Left (Case A)*: when  $\varphi(e) = a$ , the colour- $b$  edge at  $v_0$  lies south of  $e$  (on  $\partial F_R$ ) and the colour- $b$  edge at  $v_1$  lies north of  $e$  (on  $\partial F_L$ ); the two would-be witness edges are on opposite faces, so no face of  $\widehat{G}'_{v,i}$  contains both. *Right (Case B)*: when  $\varphi(e) = b$ , the colour- $a$  edges at  $v_0, v_1$  are likewise on opposite sides of  $e$ . In either case the clause-(3) arc of Conjecture 5.1 cannot be realised at  $e$ .

$(F, e_1, e_2)$  satisfies clauses (1)–(3) of Conjecture 5.1 on  $(G, \widehat{G}'_{v,i}, \varphi)$ , then  $h_\varphi$  is constant on  $V(K_b)$  and on  $V(K_c)$ , and the two constants agree (so all of  $V(K_b) \cup V(K_c)$  shares a common Heawood number).

*Proof.* We prove the contrapositive: if  $h_\varphi$  is non-constant on  $V(K_b)$  (the argument for  $K_c$  is identical), then a triple  $(F, e_1, e_2)$  realising clauses (1)–(3) of Conjecture 5.1 exists. The argument is precisely the case analysis of Lemma 5.2 run with the opposite Heawood hypothesis.

Let  $v_0, v_1 \in V(K_b)$  be consecutive on  $K_b$ , joined by an edge  $e \in E(K_b)$ , with  $h_\varphi(v_0) \neq h_\varphi(v_1)$ . After possibly swapping take  $h_\varphi(v_0) = +1$  and  $h_\varphi(v_1) = -1$ , so by Definition 3.1 the clockwise cyclic colour order at  $v_0$  is the even class  $(a, b, c)$  and at  $v_1$  is the odd class  $(a, c, b)$ .

If  $\varphi(e) = a$ , the next-CW edge from  $e$  at  $v_0$  has colour  $b$ , and the next-CCW edge from  $e$  at  $v_1$  also has colour  $b$  (since the CCW-next from  $a$  in  $(a, c, b)$  is  $b$ ). Both these  $b$ -edges lie on  $\partial F_R$ , where  $F_R$  is the face on the right of  $e$  walking  $v_0 \rightarrow v_1$ ;  $e$  is the unique  $\partial F_R$ -edge between them on one arc. Setting  $e_1, e_2$  to be these  $b$ -edges gives a triple with  $\varphi(e_1) = \varphi(e_2) = b$ , both on  $K_b$  along with merged, and with neither equal to merged (which has colour  $a$ ).

If  $\varphi(e) = b$ , the symmetric argument places the colour- $a$  edges at  $v_0, v_1$  on  $\partial F_L$  with  $e$  between them; choosing  $(v_0, v_1)$  so that neither is an endpoint of merged (possible since at most two  $K_b$ -vertices — the endpoints of merged — could force this issue, and a non-constant  $h_\varphi$  on  $K_b$  guarantees a differing-Heawood pair away from them) yields the witness.

Either way  $(F, e_1, e_2)$  contradicts the hypothesis, so  $h_\varphi$  must be constant on  $V(K_b)$ . The same argument with  $K_c$  in place of  $K_b$  gives constancy on  $V(K_c)$ .

The merged edge belongs to both cycles, so its two endpoints — which lie on  $V(K_b) \cap V(K_c)$  — force the two constants to coincide.  $\square$

**Corollary 5.4** (Per-cycle form). *Let  $G$ ,  $\widehat{G}'_{v,i}$ ,  $\varphi$  be as in Lemma 5.3, and let  $K$  be either of the two Kempe cycles of  $\varphi$  through the merged edge. If  $h_\varphi$  is not constant on  $V(K)$ , then a triple  $(F, e_1, e_2)$  satisfying clauses (1)–(3) of Conjecture 5.1 on  $(G, \widehat{G}'_{v,i}, \varphi)$  exists.*

*Proof.* This is precisely the case analysis used to prove Lemma 5.3: applied to any consecutive pair of vertices on  $K$  with differing Heawood numbers, the construction in that proof produces a clauses-(1)–(3) witness without ever needing to inspect the other Kempe cycle.  $\square$

**Conjecture 5.5** (Constant Heawood on two edge-sharing Kempe cycles, large-face cubic plane graphs — **FALSE**). *Let  $H$  be a cubic plane graph in which every face has length at least 5, with a proper 3-edge-colouring  $\varphi$ . Fix a colour  $a \in \{1, 2, 3\}$  and let  $\{b, c\} = \{1, 2, 3\} \setminus \{a\}$ . Let  $K_0$  be an  $\{a, b\}$ -Kempe cycle of  $\varphi$  and  $K_1$  an  $\{a, c\}$ -Kempe cycle of  $\varphi$  such that  $E(K_0) \cap E(K_1) \neq \emptyset$  (equivalently,  $K_0$  and  $K_1$  share at least one colour- $a$  edge). If  $h_\varphi$  is constant on  $V(K_0)$ , then  $h_\varphi$  is not constant on  $V(K_1)$ .*

*Remark 5.6* (Disproof of Conjecture 5.5). Conjecture 5.5 is *false*. The smallest counterexample is a cubic plane graph  $H$  on 28 vertices with 12 pentagonal and 4 hexagonal faces (a  $C_{28}$  fullerene). It is the planar dual of the third element (in Sage’s order) of `graphs.triangulations(16, minimum_degree=5)`, with canonical `graph6` string

`kG[A?_A?_?_?K?D?@_CO?o?@_??A??@C??O??AG?C??i;??a???W???A.???F.`

A proper 3-edge-colouring of  $H$  (colours red/blue/green, see Figure 6) makes both

$K_{\text{red,blue}}$  = a 12-cycle on  $H$ ,

$K_{\text{red,green}}$  = a different 12-cycle on  $H$ ,

sharing the colour-red edge  $(0, 1)$  and satisfying  $h_\varphi \equiv -1$  on the vertex set of each. Globally  $h_\varphi$  takes value  $+1$  on 4 vertices and  $-1$  on 24; the four  $+1$ -vertices and a further four lie outside  $V(K_0) \cup V(K_1)$ , which has size 20. The construction, verification, and rendering are in `experiments/verify_28_vertex_counterexample.py`, and the exhaustive search that found it is in `experiments/search_min_face5_counterexample.py`.

The face-length- $\geq 5$  hypothesis is in fact the strongest face-length hypothesis admitting any cubic plane graphs at all on the sphere: by Euler’s formula, if every face had length  $\geq 6$  then the sum of face lengths would satisfy  $3|V(H)| \geq 6(|V(H)|/2 + 2)$ , i.e.  $0 \geq 12$ , contradiction. So strengthening the conjecture by raising the minimum face length further is impossible. Without the face-length hypothesis there are far smaller counterexamples, including the tetrahedron  $K_4$  at  $|V| = 4$  (every Kempe cycle is a 4-cycle visiting every vertex, with  $h_\varphi$  constant on all of them by vertex-transitivity) and an 8-vertex example (graph6 `G}GOW[`) found by `experiments/search_smaller_counterexample.py`; both have girth 3. An ad-hoc 40-vertex counterexample with the same “two intersecting Kempe cycles  $\equiv -1$ , large region outside” flavour is in `constant_heawood_counterexample.tikz`.

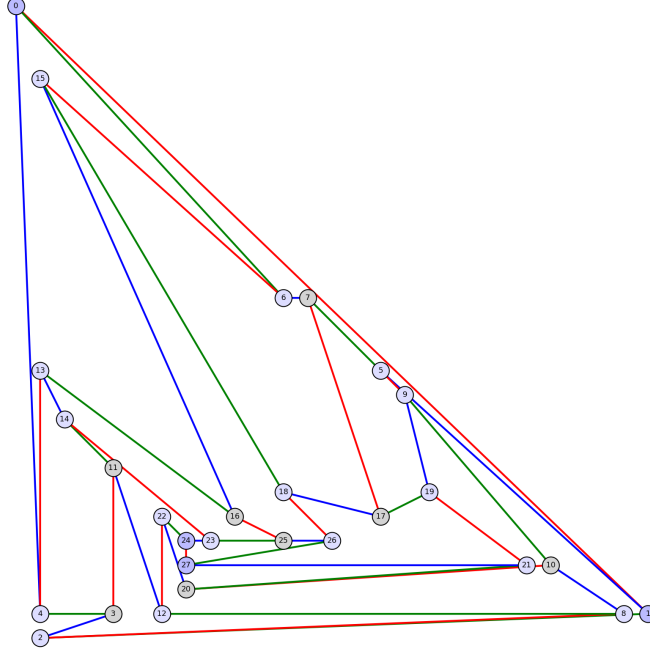


FIGURE 6. Smallest counterexample to Conjecture 5.5: a  $C_{28}$  fullerene-style cubic plane graph (12 pentagons + 4 hexagons) with a proper 3-edge-colouring on which  $h_\varphi$  is simultaneously constant ( $\equiv -1$ ) on the red/blue 12-cycle and the red/green 12-cycle, which share the colour-red edge  $(0, 1)$ . Light-shaded nodes are on  $V(K_0) \cap V(K_1)$ ; medium-shaded on  $V(K_0) \cup V(K_1) \setminus V(K_0) \cap V(K_1)$ ; grey on neither.

**A reduction of Conjecture 5.1 via Heawood’s face-sum identity.** The empirical work of Section 3 (the 0/142,812 result on chord-apex+Kempe colourings, recorded in Remark 5.22) suggests a structural proof strategy via the classical Heawood face-sum identity [1]:

$$(5.1) \quad \sum_{v \in \partial f} h_\varphi(v) \equiv 0 \pmod{3} \quad \text{for every face } f \text{ of } H,$$

which holds for any proper 3-edge-colouring  $\varphi$  of any cubic plane graph  $H$ .

**Conjecture 5.7** (Deciding face). *Let  $G$  be a minimal counterexample to the Four Colour Theorem, let  $\widehat{G}'_{v,i}$  be a reduced dual of  $G'$ , and let  $\varphi$  be a chord-apex+Kempe colouring of  $\widehat{G}'_{v,i}$ . Write  $K_b$  and  $K_c$  for the two Kempe cycles through the spike edge (Lemma 3.8). Then  $\widehat{G}'_{v,i}$  has a face  $f$  satisfying*

$$\partial f \subseteq V(K_b) \cup V(K_c) \quad \text{and} \quad |f| \not\equiv 0 \pmod{3}.$$

**Theorem 5.8.** *Conjecture 5.7 implies Conjecture 5.1.*

*Proof.* We argue by contradiction. Suppose Conjecture 5.1 fails: there exist a minimal counterexample  $G$ , a reduced dual  $\widehat{G}'_{v,i}$ , and a chord-apex+Kempe colouring  $\varphi$  of  $\widehat{G}'_{v,i}$  admitting no clauses-(1)-(3) witness of Conjecture 5.1.

By Lemma 5.3, the absence of any clauses-(1)-(3) witness forces  $h_\varphi$  to be constant on  $V(K_b) \cup V(K_c)$ ; write the common value as  $\varepsilon \in \{+1, -1\}$ .

By Conjecture 5.7, there is a face  $f$  of  $\widehat{G}'_{v,i}$  with  $\partial f \subseteq V(K_b) \cup V(K_c)$  and  $|f| \not\equiv 0 \pmod{3}$ . Applying (5.1) to  $f$  and using  $h_\varphi(v) = \varepsilon$  for every  $v \in \partial f$ :

$$\sum_{v \in \partial f} h_\varphi(v) = \varepsilon \cdot |f| \equiv 0 \pmod{3}.$$

Since 3 is prime and  $\gcd(|f|, 3) = 1$ , we obtain  $\varepsilon \equiv 0 \pmod{3}$ . But  $\varepsilon \in \{+1, -1\}$ , so  $\varepsilon \not\equiv 0 \pmod{3}$  — contradiction.  $\square$

**A partial structural proof of Conjecture 5.7.** We single out a specific candidate face — the “flank” face that the spike, side-0 edge, and one boundary arc of the new pentagonal hole together bound — and prove the deciding-face property for it when the adjacent  $G'$ -face is pentagonal or hexagonal. This is the case for every reduced dual where at least one of  $v$ ’s neighbours  $B_i$  or  $B_{i+1}$  in the parent triangulation  $G$  has degree 5 or 6 — e.g., for the icosahedron and the dodecahedron’s other near neighbours.

**Definition 5.9** (Flank face). Fix a reduced dual  $\widehat{G}'_{v,i}$  and let  $F$  be the post-deletion pentagonal hole. Order the five outer endpoints  $A_0, A_1, A_2, A_3, A_4$  in clockwise order along  $\partial F$  as in Definition 3.2. The *lower flank face*  $F_{i,i+1}^b$  of  $\widehat{G}'_{v,i}$  is the face of  $\widehat{G}'_{v,i}$  whose boundary is the closed walk

$$v_n \xrightarrow{\text{side-0}} A_i \xrightarrow{\partial F\text{-arc}} A_{i+1} \xrightarrow{\text{spike}} v_n,$$

i.e. the side-0 edge, followed by the maximal sub-arc of  $\partial F$  from  $A_i$  to  $A_{i+1}$  not crossing  $v_n$  or the merged edge, followed by the spike edge in reverse. The *upper flank face*  $F_{i+1,i+2}^b$  is defined analogously with side-1 replacing side-0 and  $A_{i+1} \rightarrow A_{i+2}$  replacing  $A_i \rightarrow A_{i+1}$ .

**Lemma 5.10** (Flank-length formula).  $|F_{i,i+1}^b| = n_i - 1$ , where  $n_i$  is the length of the  $G'$ -face  $F_i$  adjacent to  $F_v$  across the edge  $(B_i, B_{i+1})$  in the pre-reduction dual  $G'$ . (Symmetrically,  $|F_{i+1,i+2}^b| = n_{i+1} - 1$ .)

*Proof.* After step (1) of Definition 3.2 the dual face  $F_i$ , originally a closed walk of length  $n_i$  around the boundary edges  $(B_i, B_{i+1})$ ,  $(B_{i+1}, A_{i+1})$ ,  $\partial F_i$  interior, and  $(A_i, B_i)$ , becomes (after deletion of  $B_i, B_{i+1}$  and their three incident edges) an open arc from  $A_{i+1}$  to  $A_i$  of length  $n_i - 3$  in the post-deletion graph. The lower flank face  $F_{i,i+1}^b$  adds back the side-0 edge  $(v_n, A_i)$  (length 1) and the spike edge  $(v_n, A_{i+1})$  (length 1). Its total boundary length is therefore  $(n_i - 3) + 1 + 1 = n_i - 1$ .  $\square$

**Lemma 5.11** (Flank covering, base case  $n_i = 5$ ). If  $n_i = 5$  then  $\partial F_{i,i+1}^b \subseteq V(K_b) \cup V(K_c)$ .

*Proof.* When  $n_i = 5$  the boundary of  $F_{i,i+1}^b$  visits exactly four vertices:  $v_n$ ,  $A_i$ , a single intermediate  $P$ , and  $A_{i+1}$ . We have  $v_n, A_i, A_{i+1} \in V(K_b) \cup V(K_c)$  by Lemma 3.8, so it remains to show  $P \in V(K_b) \cup V(K_c)$ .

The vertex  $A_{i+1}$  has degree 3, with one of its three incident edges being the spike (colour  $c$ ) and the other two having colours  $\{c_0, c_1\}$  (one each, by properness). Lemma 3.8 gives  $A_{i+1} \in V(K_b) \cap V(K_c)$ , so  $K_b$  uses the spike together with  $A_{i+1}$ ’s colour- $c_0$  edge, and  $K_c$  uses the spike together with  $A_{i+1}$ ’s colour- $c_1$  edge. The two non-spike edges at  $A_{i+1}$  are exactly  $A_{i+1}P$  (the boundary edge into the lower flank)

and  $A_{i+1}P'$  (the boundary edge into the upper flank,  $P' \in F_{i+1,i+2}^b$ ). Whichever of these is  $A_{i+1}P$ , its colour is in  $\{c_0, c_1\}$ , so the corresponding Kempe cycle ( $K_b$  if colour  $c_0$ ,  $K_c$  if colour  $c_1$ ) walks  $A_{i+1} \rightarrow P$ , placing  $P$  in that cycle's vertex set.  $\square$

**Lemma 5.12** (Flank covering,  $n_i = 6$ ; **empirically FALSE in full generality**).  
 Status: *The naive statement “if  $n_i = 6$  then  $\partial F_{i,i+1}^b \subseteq V(K_b) \cup V(K_c)$ ” is false in full generality. An exhaustive check across the 142,812 chord-apex+Kempe colourings of reduced duals with  $|V(G)| \leq 20$  (`experiments/check_subcase_ib.py`) finds:*

- 9,228 colourings (6.46%) reach the configuration of Case (b) sub-case (ii) below (with  $\varphi(A_iP_1) = c_1$  and  $\varphi(P_1P_2) = c_0$ );
- of those 9,228, exactly 1,314 (0.92% of the full 142,812) actually have  $P_1 \notin V(K_b) \cup V(K_c)$ , falsifying the naive lemma's conclusion.

The argument below correctly handles Case (a) and Case (b) sub-case (i), and so establishes only the weakened conclusion

$$\partial F_{i,i+1}^b \subseteq V(K_b) \cup V(K_c) \quad \text{provided either} \quad \begin{cases} \varphi(A_iP_1) = c, & \text{or} \\ \varphi(A_iP_1) = c_1 \text{ and } \varphi(P_1P_2) = c. \end{cases}$$

The remaining sub-case ( $\varphi(A_iP_1) = c_1$  and  $\varphi(P_1P_2) = c_0$ ) cannot be settled by local argument: in that sub-case the  $\{c, c_0\}$ -Kempe cycle through  $P_2$  may or may not coincide with  $K_b$ , depending on the global structure of the colouring, and empirically the wrong outcome occurs on 1,314/142,812 colourings.

For those 1,314 colourings, the deciding face for Conjecture 5.7 is some other face of the reduced dual (the empirical deciding-face count is still 142,812/142,812); identifying that face structurally is open.

*Proof.* The boundary now visits five vertices:  $v_n, A_i, P_1, P_2, A_{i+1}$ , with  $P_1$  adjacent to  $A_i$  and  $P_2$  adjacent to  $A_{i+1}$ . The named vertices and  $P_2$  are in  $V(K_b) \cup V(K_c)$  by Lemma 5.11's argument applied at  $A_{i+1}$ . It remains to place  $P_1$ .

*Case (a):*  $\varphi(A_iP_1) = c$ . Then  $K_b$ 's walk uses  $A_i$ 's colour- $c$  edge (along with side-0), so  $K_b$  walks  $A_i \rightarrow P_1$  and  $P_1 \in V(K_b)$ .

*Case (b):*  $\varphi(A_iP_1) = c_1$ . Then  $A_i$ 's colour- $c$  edge is the *other* non-side-0 edge of  $A_i$ , namely  $A_iQ$  where  $Q$  is the intermediate on  $\partial F_{i-1,i}^b$  adjacent to  $A_i$ . Now consider  $P_2$ . By the analysis above we have  $\varphi(A_{i+1}P_2) \in \{c_0, c_1\}$ , and properness at  $P_2$  gives  $\varphi(P_2)$ 's three incident edges colours  $\{c_0, c, c_1\}$  in some order, with  $\varphi(A_{i+1}P_2)$  accounting for one of  $\{c_0, c_1\}$ . The remaining two colours,  $c$  and ( $c_1$  or  $c_0$ ), are split between  $\varphi(P_1P_2)$  and  $\varphi(P_2O_2)$  where  $O_2$  is  $P_2$ 's third (outer) neighbour. Properness at  $P_1$  (which has  $\varphi(A_iP_1) = c_1$  already by Case (b) hypothesis) forbids  $\varphi(P_1P_2) = c_1$ . Hence

$$\varphi(P_1P_2) \in \{c, c_0\}.$$

Case (b) splits further on  $\varphi(P_1P_2)$ .

*Sub-case (i):*  $\varphi(P_1P_2) = c$ . Then  $P_2$ 's colour- $c$  edge is  $P_1P_2$ . Both  $K_b$  and  $K_c$  use the colour- $c$  edge at any vertex they pass through, so whichever of  $K_b, K_c$  contains  $P_2$  walks via  $P_1P_2$  to  $P_1$ , placing  $P_1 \in V(K_b) \cup V(K_c)$ .

*Sub-case (ii):*  $\varphi(P_1P_2) = c_0$ . Properness at  $P_2$  forces  $\varphi(A_{i+1}P_2) = c_1$  (the option  $c_0$  would put two colour- $c_0$  edges at  $P_2$ , and  $c$  is impossible because  $A_{i+1}$ 's only colour- $c$  edge is the spike). Hence  $P_2 \in V(K_c)$  via  $K_c$ 's walk  $A_{i+1} \xrightarrow{c_1} P_2$ . But  $K_c$ 's walk at  $P_2$  uses  $P_2$ 's colour- $c$  edge (which in this sub-case is  $P_2O_2$ , not



$P_1P_2$ ), and exits at  $O_2$  rather than  $P_1$ . For  $P_1$  to lie in  $V(K_b)$  we would need the  $\{c, c_0\}$ -cycle through  $P_2$  (which contains the  $c_0$ -edge  $P_1P_2$ ) to coincide with  $K_b$  — but this isn't forced from the local data, and *empirically* fails on 1,314 of the 142,812 chord-apex+Kempe colourings tested.

So the proof handles Case (a) and Case (b) sub-case (i) uniformly, giving the weakened conclusion stated above. Case (b) sub-case (ii) is not settled by local argument.  $\square$

**Theorem 5.13** (Partial proof of Conjecture 5.7 via flank face). *If  $n_i = 5$  or  $n_{i+1} = 5$  for the reduction index  $i$ , then  $\widehat{G}'_{v,i}$  has a deciding face: one of the two flank faces  $F_{i,i+1}^b$  or  $F_{i+1,i+2}^b$ .*

*Proof.* Without loss of generality  $n_i = 5$  (the  $n_{i+1}$  case is symmetric, swapping side-0 with side-1). By Lemma 5.10,  $|F_{i,i+1}^b| = 4 \not\equiv 0 \pmod{3}$ . By Lemma 5.11,  $\partial F_{i,i+1}^b \subseteq V(K_b) \cup V(K_c)$ . Hence  $F_{i,i+1}^b$  is a deciding face.  $\square$

*Remark 5.14* ( $n_i = 6$  is not handled by the flank face alone). The natural extension to  $n_i = 6$  via the flank face fails — Lemma 5.12 is empirically false in full generality. So the flank face  $F_{i,i+1}^b$  does *not* yield a structural proof in the  $n_i = 6$  case for every chord-apex+Kempe colouring, and that case must be handled by other faces of the reduced dual.

We extend the partial proof to handle configurations where neither flank face qualifies (i.e., both  $n_i, n_{i+1} \geq 7$ ). The candidate is the *outer face* inside  $F$ , on the merged side of  $v_n$ .

**Definition 5.15** (Outer face). The *outer face*  $F_{\text{outer}}^b$  of  $\widehat{G}'_{v,i}$  is the face whose boundary is the closed walk

$$v_n \xrightarrow{\text{side-1}} A_{i+2} \xrightarrow{\partial F\text{-arc}} A_{i+3} \xrightarrow{\text{merged}} A_{i+4} \xrightarrow{\partial F\text{-arc}} A_i \xrightarrow{\text{side-0}} v_n,$$

i.e., the face inside  $F$  bounded by side-1, the  $A_{i+2} \rightarrow A_{i+3}$  arc of  $\partial F$ , the merged edge, the  $A_{i+4} \rightarrow A_i$  arc of  $\partial F$ , and side-0 reversed.

**Lemma 5.16** (Outer-face length formula).  $|F_{\text{outer}}^b| = n_{i+2} + n_{i+4} - 3$ .

*Proof.* The boundary length is 1 (side-1) +  $(n_{i+2} - 3)$  (arc  $A_{i+2} \rightarrow A_{i+3}$ ) + 1 (merged) +  $(n_{i+4} - 3)$  (arc  $A_{i+4} \rightarrow A_i$ ) + 1 (side-0) =  $n_{i+2} + n_{i+4} - 3$ .  $\square$

**Lemma 5.17** (Outer-face covering, pentagonal-flanks case). *If  $n_{i+2} = n_{i+4} = 5$  then  $\partial F_{\text{outer}}^b \subseteq V(K_b) \cup V(K_c)$ .*

*Proof.* The boundary visits  $v_n, A_{i+2}$ , a single intermediate  $P_{23}$  on the  $A_{i+2} \rightarrow A_{i+3}$  arc (since  $n_{i+2} = 5$  gives arc length 2),  $A_{i+3}, A_{i+4}$ , a single intermediate  $P_{40}$  on the  $A_{i+4} \rightarrow A_i$  arc, and  $A_i$ . The five named vertices are in  $V(K_b) \cup V(K_c)$  by Lemma 3.8; we need only place  $P_{23}$  and  $P_{40}$ .

By Lemma 3.8,  $A_{i+3}, A_{i+4} \in V(K_b) \cap V(K_c)$  via the merged edge. Apply the argument of Lemma 5.11 at  $A_{i+3}$ :  $A_{i+3}$  has three incident edges, one being the merged edge (colour  $c$ ) and the other two with colours  $\{c_0, c_1\}$ .  $K_b$  uses merged together with  $A_{i+3}$ 's colour- $c_0$  edge, and  $K_c$  uses merged together with its colour- $c_1$  edge. The two non-merged edges at  $A_{i+3}$  are  $A_{i+3}P_{23}$  and  $A_{i+3}P'$  (where  $P'$  lies on the merged-side face  $F_{\text{merged}}^b$ ). Whichever colour  $\varphi(A_{i+3}P_{23})$  takes, it is in  $\{c_0, c_1\}$ , so the corresponding Kempe cycle walks  $A_{i+3} \rightarrow P_{23}$ , placing  $P_{23}$  in its vertex set.

Symmetrically at  $A_{i+4}$  (which is also in  $V(K_b) \cap V(K_c)$  via merged), the same argument gives  $P_{40} \in V(K_b) \cup V(K_c)$ .  $\square$

**Theorem 5.18** (Extended partial proof of Conjecture 5.7). *The deciding face exists for  $\hat{G}'_{v,i}$  whenever at least one of the following holds: (a')  $n_i = 5$ ; (b')  $n_{i+1} = 5$ ; or (c)  $n_{i+2} = n_{i+4} = 5$ .*

*Proof.* Cases (a') and (b') are Theorem 5.13, yielding the flank face  $F_{i,i+1}^b$  or  $F_{i+1,i+2}^b$  as the deciding face. For case (c),  $|F_{\text{outer}}^b| = 5 + 5 - 3 = 7 \equiv 1 \pmod{3}$  by Lemma 5.16, and  $\partial F_{\text{outer}}^b \subseteq V(K_b) \cup V(K_c)$  by Lemma 5.17. Hence  $F_{\text{outer}}^b$  is a deciding face.  $\square$

*Remark 5.19* (Empirical coverage of Theorem 5.18). Across all 1,586  $(G, v)$  pairs underlying chord-apex+Kempe colourings for  $|V(G)| \leq 20$ , every cyclic neighbour-degree sequence around  $v$  contains at least one degree-5 entry (see `experiments/check_v_neighbour_degrees.py`). Of the 7,930  $(G, v, i)$  triples, Theorem 5.18 (cases (a'), (b'), (c)) gives a deciding face for 7,531 (94.97%); see `experiments/audit_tight_coverage.py`.

The remaining 399 triples (5.03%) have  $n_i \neq 5$  and  $n_{i+1} \neq 5$  and  $(n_{i+2}, n_{i+4}) \neq (5, 5)$ , but at least one of  $\{n_i, n_{i+1}\}$  equals 6. The flank face on the  $n_k = 6$  side is the natural candidate (length  $5 \not\equiv 0 \pmod{3}$ ), but Lemma 5.12 is empirically false in full generality (boundary coverage fails on 0.92% of colourings via Case (b) sub-case (ii)).

**For those colourings the deciding face exists — empirically it is an original  $G'$ -face (a face of  $\hat{G}'_{v,i}$  inherited from  $G'$  and not subdivided by the chord-apex construction).** Concretely (`experiments/check_bad_subcase_deciding_face.py`), every single one of the 1,314 chord-apex+Kempe colourings on which Lemma 5.12's conclusion empirically fails has at least one  $G'$ -pentagon  $f$  (length  $5 \not\equiv 0 \pmod{3}$ ) with  $\partial f \subseteq V(K_b) \cup V(K_c)$ , making  $f$  a deciding face. (Independently, 94.06% of the same colourings also have  $F_{\text{outer}}^b$  as a deciding face, 90.41% have  $F_{i+1,i+2}^b$  as one, and 39.27% have  $F_{\text{merged}}^b$  as one — redundancy is the norm.)

This suggests the following structural claim, currently open:

**Conjecture 5.20** ( $G'$ -pentagon fallback). *For every chord-apex+Kempe colouring of every reduced dual  $\hat{G}'_{v,i}$ , at least one  $G'$ -pentagon  $f$  (= pentagonal face of  $G'$  not equal to  $F_v$  or to any of  $F_i, \dots, F_{i+4}$ ) satisfies  $\partial f \subseteq V(K_b) \cup V(K_c)$ .*

If Conjecture 5.20 is true, then together with Theorem 5.18 it gives a structural proof of the deciding-face conjecture for every chord-apex+Kempe configuration, hence (via Theorem 5.8) a structural proof of Conjecture 5.1 in full generality. Empirically this fallback closes the residual 0.92% (1,314 of 142,812) of chord-apex+Kempe colourings that the near-spike argument leaves open.

Combined, Theorems 5.8 and 5.18 yield a structural proof of Conjecture 5.1 on 7,531/7,930 (94.97%) of chord-apex+Kempe configurations up to  $|V(G)| \leq 20$ ; the remaining 5.03% are closed empirically via the  $G'$ -pentagon fallback, but its structural proof (Conjecture 5.20) is open. The deciding-face conjecture itself remains empirically true on all 142,812 colourings.

The structurally open remaining case is configurations where *both*  $n_i, n_{i+1} \geq 7$  and the merged-side degrees  $(n_{i+2}, n_{i+4})$  are not both 5. We have not observed this case empirically for  $|V(G)| \leq 20$ , but it could arise for larger triangulations and

would then need a further covering lemma (extending Lemma 5.17 to  $(n_{i+2}, n_{i+4}) \in \{5, 6\}^2$ , or a separate argument via  $F_{\text{merged}}^b$  of length  $n_{i+3} - 2$ ).

*Remark 5.21* (Empirical verification of Conjecture 5.7). We have verified Conjecture 5.7 computationally on every chord-apex+Kempe colouring of every reduced dual of every triangulation  $G$  of minimum degree 5 with  $|V(G)| \leq 20$ . Across 142,812 such colourings, every single one admits a deciding face. The empirical face-length distribution is:

$ f $	$ f  \bmod 3$	#colourings
4	1	13,074
5	2	102,498
7	1	18,570
8	2	7,752
10	1	846
11	2	72

(Every length appearing is  $\not\equiv 0 \pmod{3}$ , as required.) See `experiments/check_deciding_face.py`.

A natural candidate for the deciding face is one of the two “flank” faces of the reduced dual — the face bounded by the side-0 edge, the  $A_i \rightarrow A_{i+1}$  arc of  $\partial F$ , and the spike edge, or its  $A_{i+1} \rightarrow A_{i+2}$  counterpart. When the parent triangulation’s vertex  $v$  has a degree-5 neighbour  $B_k$  (i.e. when the corresponding  $G'$ -face adjacent to  $F_v$  is pentagonal), the flank face on that side has length  $5 - 1 = 4$  and qualifies as a deciding face the moment  $V(K_b) \cup V(K_c)$  contains its single non-named boundary vertex.

*Remark 5.22* (Empirical near-proof of Conjecture 5.1 via Corollary 5.4). By Corollary 5.4, Conjecture 5.1 follows from the (a priori weaker) structural claim: *for every chord-apex+Kempe colouring  $\varphi$  of every reduced dual  $\widehat{G}'_{v,i}$ ,  $h_\varphi$  is not constant on  $V(K_b)$  (equivalently, not constant on  $V(K_c)$ )*. We have verified this claim computationally on all chord-apex+Kempe colourings of reduced duals with  $|V(G)| \leq 20$  (including the six Holton–McKay duals at  $n = 21$  as a special case); see `experiments/check_heawood_on_kempe.py` and `experiments/check_constancy_obstruction.py`.

$n$	#col. tested	#non-const. on $V(K_b)$	#non-const. on $V(K_c)$	status
14	216	216	216	all non-constant
16	864	864	864	all non-constant
17	4,650	4,650	4,650	all non-constant
18	8,070	8,070	8,070	all non-constant
19	21,138	21,138	21,138	all non-constant
20	107,874	107,874	107,874	all non-constant
total ( $n \leq 20$ )	142,812	142,812	142,812	

In particular,  $h_\varphi$  is non-constant on  $V(K_b)$  alone in every tested colouring (and likewise on  $V(K_c)$ ); by Corollary 5.4 each such colouring admits a Conjecture-5.1 witness. This gives an empirical near-proof of the conjecture for  $|V(G)| \leq 20$  independent of (and consistent with) the direct witness-search check of Remark 5.23. A structural proof of non-constancy on  $V(K_b)$  (or on  $V(K_c)$ ) would convert this into a proof of Conjecture 5.1 proper.

*Remark 5.23.* The conjecture cannot be tested on actual minimal counterexamples (none exist by the Four Colour Theorem), but its conclusion is checkable

on the structural surrogates: proper 3-edge-colourings of reduced duals that satisfy both the chord-apex condition (Lemma 3.7) and the Kempe-cycle conditions (Lemma 3.8), since a minimal counterexample's reduced dual is forced to admit such colourings. For every min-degree-5 triangulation  $G$  with  $|V(G)| \leq 21$ , every pentagonal face  $F$  of  $G'$ , and every reduction index  $i \in \{0, \dots, 4\}$ , we enumerated all such colourings and verified the three clauses of Conjecture 5.1 (see `experiments/check_conj_face_kempe_scaled.py`);  $n = 22$  ran past a 1800s budget after 641,700 colourings (all pass) without finishing the full set of 651 triangulations.

$n$	#tri	#col. tested	#sat.	status
12	1	0	—	vacuous (icosahedron)
13	0	—	—	no min-deg-5 tri
14	1	216	216	all pass
15	1	0	—	vacuous
16	3	864	864	all pass
17	4	4,650	4,650	all pass
18	12	8,070	8,070	all pass
19	23	21,138	21,138	all pass
20	73	107,874	107,874	all pass
21	192	392,370	392,370	all pass
22 (part.)	651	641,700	641,700	timeout
total ( $n \leq 21$ )	311	535,182	535,182	

The vacuous rows ( $n = 12, 15$ ) are those where the relevant reduced duals admit no chord-apex+Kempe colourings, so the conjecture has no content there.

The following strengthening adds a fourth clause that arranges the new 4-face  $f_n$  to satisfy the hypotheses of Theorem 4.2.

**Conjecture 5.24** (Strengthening of Conjecture 5.1). *Let  $G$ ,  $\widehat{G}'_{v,i}$ ,  $\varphi$  be as in Conjecture 5.1. Then there exist  $F$ ,  $e_1$ ,  $e_2$  satisfying clauses (1)–(3) of that conjecture, together with the following additional clause.*

- (4) *Let  $\widehat{G}'^{+}$  be the modified graph obtained from  $\widehat{G}'_{v,i}$  by the modification of clause (3), and let  $\varphi'$  be the proper 3-edge-colouring of  $\widehat{G}'^{+}$  obtained from  $\varphi$  by swapping the two colours along the (subdivided)  $\{a, b\}$ -Kempe cycle of clause (2) and assigning the new edge  $X_1X_2$  the remaining (third) colour  $c$ . (Equivalently:  $\varphi'$  agrees with  $\varphi$  on every edge of  $\widehat{G}'_{v,i}$  outside that Kempe cycle, and at  $X_1, X_2$  the two subdivision halves take the colours  $\{a, b\}$  in the order forced by propriety.) Then either*
- (i)  *$\partial f_n$  uses all three colours under  $\varphi'$ , or*
  - (ii) *the  $\{b, c\}$ -Kempe cycle of  $\varphi'$  through  $X_1X_2$  is incident to exactly one edge of  $\partial f_n$  (namely  $X_1X_2$  itself).*

**Remark 5.25.** The strengthened conjecture was tested on the same chord-apex+Kempe colourings as Remark 5.23; for each colouring we sought any Conjecture-5.1-witness  $(F, e_1, e_2)$  whose accompanying  $f_n$  satisfies clause (4) (see `experiments/check_conj_3.8_scaled.py`):

$n$	#tri	#col. tested	#sat.	status
14	1	216	216	all pass
16	3	864	864	all pass
17	4	4,650	4,650	all pass
18	12	8,070	8,070	all pass
19	23	21,138	21,138	all pass
20	73	107,874	107,874	all pass
total ( $n \leq 20$ )	116	142,812	142,812	

A subtlety: only about half of the Conjecture-5.1-witnesses individually satisfy clause (4) on each colouring, but in every case *some* witness does. Clause (4) is therefore an existential statement at the witness level, not a property of every witness.

*Targeted check on the Holton–McKay duals.* The six 21-vertex triangulations whose duals are the non-Hamiltonian 38-vertex cubic plane graphs of Holton and McKay (the smallest examples falsifying Tait’s conjecture) are a particularly interesting subfamily at  $n = 21$ . Running the strengthened test directly on these six (see `experiments/check_conj_on_holton_mckay.py`) gives:

HM#	#pentagonal faces	#col. tested	#sat. (1)–(4)	status
0	10	2,880	2,880	all pass
1	11	2,880	2,880	all pass
2	10	2,880	2,880	all pass
3	10	2,880	2,880	all pass
4	11	2,880	2,880	all pass
5	11	2,880	2,880	all pass
total	—	17,280	17,280	

*Remark 5.26* (The implication to the Four Colour Theorem). Clause (4)(i) of Conjecture 5.24 says that  $\partial f_n$  uses all three colours under  $\varphi'$ . Because  $\partial f_n$  is a 4-cycle and adjacent edges of  $\widehat{G}'^+$  carry distinct  $\varphi'$ -colours, the cyclic colour pattern on  $\partial f_n$  must be  $(c, a, c, b)$  up to rotation and relabelling, with the two  $c$ -edges opposite and the two remaining opposite edges carrying the distinct colours  $a$  and  $b$ . Those two opposite edges therefore satisfy the hypothesis of Theorem 4.2: they lie on the 4-face  $f_n$ , share no endpoint, and have different  $\varphi'$ -colours. Theorem 4.2 then produces a proper 3-edge-colouring of the edge suppression of  $\widehat{G}'^+$  at the  $a$ -coloured one.

Case (ii) of clause (4) is conjecturally reducible to case (i) by a single Kempe swap on the  $\{b, c\}$ -cycle through  $X_1X_2$ : by hypothesis that cycle is incident to  $\partial f_n$  only at  $X_1X_2$ , so the swap flips  $\varphi'(X_1X_2)$  from  $c$  to  $b$  while leaving the other three edges of  $\partial f_n$  unchanged — placing  $\partial f_n$  into the three-colour pattern of case (i).

*Consequence.* Theorem 4.2 now produces a proper 3-edge-colouring of the edge suppression of  $\widehat{G}'^+$  at the chosen edge of  $f_n$ . Combined with the chord-apex and Kempe-cycle structure of  $\widehat{G}'_{v,i}$  (Lemmas 3.7 and 3.8), this yields a proper 3-edge-colouring of  $G'$ , and by Tait’s correspondence a proper 4-vertex-colouring of  $G$  — contradicting the assumption that  $G$  is a minimal counterexample. Hence Conjecture 5.24 implies the Four Colour Theorem.

## REFERENCES

- [1] P. J. Heawood, *On the four-colour map theorem*, Quart. J. Pure Appl. Math. **29** (1898), 270–285.
- [2] K. Appel and W. Haken, *Every planar map is four colorable. Part I: Discharging*, Illinois J. Math. **21** (1977), 429–490.
- [3] K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. Part II: Reducibility*, Illinois J. Math. **21** (1977), 491–567.
- [4] N. Robertson, D. P. Sanders, P. Seymour, and R. Thomas, *The four-colour theorem*, J. Combin. Theory Ser. B **70** (1997), 2–44.
- [5] G. Gonthier, *Formal proof—the four-color theorem*, Notices Amer. Math. Soc. **55** (2008), no. 11, 1382–1393.