

PLANE DIAMOND COLORING

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ABSTRACT.

NOTATION

For a coloring $C : V(G) \rightarrow S$ and a color $c \in S$, we write $C^{-1}(c) = \{v \in V(G) : C(v) = c\}$ for the preimage of c under C , i.e., the color class of c .

1. DEFINITIONS

Definition 1.1. Let G be a graph and let $u \in V(G)$. The *distance partition* of G from u is the partition $\{L_0, L_1, L_2, \dots\}$ of $V(G)$ obtained by breadth-first search from u :

$$L_0 = \{u\}, \quad L_{i+1} = \{v \in V(G) \setminus (L_0 \cup \dots \cup L_i) : v \text{ is adjacent to some } w \in L_i\}.$$

Equivalently, $L_i = \{v \in V(G) : d(v, u) = i\}$, where $d(v, u)$ denotes the graph distance between v and u in G . We call each L_i the *i-th level* of the partition.

Definition 1.2. Let G be a maximal planar graph with a plane embedding, and let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from some $u \in V(G)$. The *diamond scaffold* of G relative to u is the spanning subgraph $G^\diamond \subseteq G$ obtained by removing every edge $\{x, y\} \in E(G)$ such that $x, y \in L_i$ for some i .

Definition 1.3. Let G be a maximal planar graph. A *plane diamond coloring* of G is a proper 4-coloring C of G such that there exist two colors c_a, c_b and a diamond scaffold G^\diamond of G with a proper 2-coloring $C^\diamond : V(G) \rightarrow \{c_a, c_b\}$ satisfying

$$\begin{aligned} C^\diamond(v) &= c_a && \text{for every } v \in C^{-1}(c_a), \\ C^\diamond(v) &= c_b && \text{for every } v \in C^{-1}(c_b). \end{aligned}$$

2. RESULTS

Theorem 2.1. *The diamond scaffold of any maximal planar graph G is 2-colorable.*

Proof. Let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from the chosen vertex u , and let G^\diamond be the resulting diamond scaffold. We show G^\diamond is bipartite by exhibiting a proper 2-coloring.

For any edge $\{x, y\} \in E(G)$, the depths of x and y differ by at most 1: if $x \in L_i$, then prepending the edge $\{y, x\}$ to a shortest path from x to u gives a walk of length $i + 1$ from y to u , so $y \in L_j$ for some $j \leq i + 1$, and symmetrically $i \leq j + 1$. Hence $|i - j| \leq 1$.

By construction, G^\diamond contains no edge with both endpoints in the same level L_i . Combined with the bound above, every edge of G^\diamond joins some L_i to L_{i+1} . Color

each vertex $v \in L_i$ by the parity of i . Every edge of G^\diamond connects vertices of opposite parity, so this is a proper 2-coloring. \square

Lemma 2.2. *The diamond scaffold G^\diamond of a maximal planar graph G relative to u is connected.*

Proof. Let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from u . We show by induction on i that every vertex of L_i is connected to u in G^\diamond . The base case $i = 0$ is immediate, since $L_0 = \{u\}$. For $i \geq 1$, let $v \in L_i$. By definition of L_i , there is a shortest path from v to u of length i in G , whose penultimate vertex w lies in L_{i-1} . The edge $\{v, w\}$ joins L_i to L_{i-1} , hence is not a level edge, hence belongs to G^\diamond . By the inductive hypothesis w is connected to u in G^\diamond , so v is as well. \square

Proposition 2.3. *A maximal planar graph G has a plane diamond coloring if and only if there exist a proper 4-coloring C of G , a vertex $u \in V(G)$, and two distinct colors c_a, c_b such that, with respect to the distance partition $\{L_0, L_1, L_2, \dots\}$ of G from u ,*

$$C^{-1}(c_a) \subseteq \bigcup_{i \text{ even}} L_i \quad \text{and} \quad C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i.$$

Proof. Since G^\diamond is connected and bipartite (Theorem 2.1 and Lemma 2.2), its proper 2-coloring is unique up to swapping the two colors, and is given by the parity of level. Hence a proper 2-coloring $C^\diamond : V(G) \rightarrow \{c_a, c_b\}$ of G^\diamond exists with $C^\diamond(v) = c_a$ on the even-parity layers and $C^\diamond(v) = c_b$ on the odd-parity layers (or vice versa). The agreement condition $C(v) = C^\diamond(v)$ on $C^{-1}(c_a) \cup C^{-1}(c_b)$ is then equivalent to the stated containment. \square

Remark 2.4. The conjecture below asserts a structural property of 4-colorings of maximal planar graphs strictly stronger than the conclusion of the Four Color Theorem [1, 2]: it requires not merely the existence of a proper 4-coloring, but the existence of a proper 4-coloring together with a root u such that two of the four color classes are separated by the parity of the BFS layering from u .

Conjecture 2.5. *Every maximal planar graph G has a plane diamond coloring.*

Theorem 2.6. *Conjecture 2.5 is false.*

Proof. Let G be the maximal planar graph on 16 vertices with graph6 string¹

0????IAGKEBEQQYHdp1W{n

shown in Figure 1. Equivalently, G has edge set

$$\begin{aligned} &\{0, 8\}, \{0, 10\}, \{0, 14\}, \{1, 7\}, \{1, 10\}, \{1, 12\}, \{2, 9\}, \{2, 11\}, \{2, 13\}, \\ &\{3, 9\}, \{3, 11\}, \{3, 15\}, \{4, 8\}, \{4, 12\}, \{4, 14\}, \{4, 15\}, \{5, 6\}, \{5, 13\}, \\ &\{5, 14\}, \{5, 15\}, \{6, 11\}, \{6, 13\}, \{6, 15\}, \{7, 10\}, \{7, 12\}, \{7, 14\}, \{8, 10\}, \\ &\{8, 12\}, \{8, 14\}, \{9, 11\}, \{9, 13\}, \{9, 15\}, \{10, 12\}, \{10, 14\}, \{11, 13\}, \\ &\{11, 15\}, \{12, 13\}, \{12, 14\}, \{12, 15\}, \{13, 14\}, \{13, 15\}, \{14, 15\}. \end{aligned}$$

We have $|V(G)| = 16$ and $|E(G)| = 42 = 3 \cdot 16 - 6$, so G is a triangulation.

By Proposition 2.4, it suffices to verify that for every choice of root $u \in V(G)$ and every pair of distinct colors c_a, c_b , no proper 4-coloring C of G satisfies $C^{-1}(c_a) \subseteq$

¹We use the standard graph6 encoding of McKay; see [3].

$\bigcup_{i \text{ even}} L_i^{(u)}$ and $C^{-1}(c_b) \subseteq \bigcup_{i \text{ odd}} L_i^{(u)}$, where $\{L_i^{(u)}\}$ denotes the distance partition from u .

For a fixed root u , the existence of such a 4-coloring C together with colors c_a, c_b is equivalent to 4-colorability of the auxiliary graph H_u obtained from G by adjoining two new vertices α, β , joining α to every vertex in $\bigcup_{i \text{ odd}} L_i^{(u)}$, joining β to every vertex in $\bigcup_{i \text{ even}} L_i^{(u)}$, and adding the edge $\{\alpha, \beta\}$. Indeed, in any proper 4-coloring of H_u the colors of α and β are distinct and absent from the odd-parity and even-parity layers of G respectively, yielding the required colors $c_a := C(\alpha)$ and $c_b := C(\beta)$. Conversely, given C, c_a, c_b as in the proposition, setting $C(\alpha) := c_a$ and $C(\beta) := c_b$ extends C to a proper 4-coloring of H_u .

A direct computation (using Sage's `chromatic_number`) verifies that $\chi(H_u) > 4$ for every $u \in V(G)$, so G admits no plane diamond coloring. \square

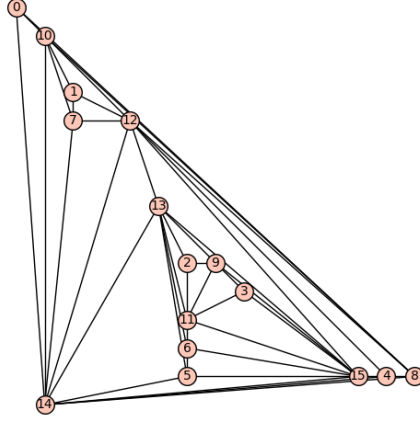


FIGURE 1. A maximal planar graph on 16 vertices with no plane diamond coloring.

REFERENCES

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- [3] B. D. McKay, *Description of graph6, sparse6 and digraph6 encodings*, <https://users.cecs.anu.edu.au/~bdm/data/formats.txt>.