COLORING NESTED TIRE GRAPHS

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Abstract. We establish the foundational structure of nested level-induced tire decompositions of a plane triangulation G. A *level source* of G induces a BFS layering of G and endows the inner planar dual G' with a *dual depth* grading. The basic object of study is the *tire graph* T — a plane graph whose outer and inner boundaries bound a closed planar region, the *tire tread* R, triangulated by the *annular edges* E_{ann}. Our main structural result, the *tire-component lemma*, exhibits each connected component of G'_d as a tire graph; the *tire-tread partition theorem* consequence shows the resulting tire treads partition the bounded faces of G. Coloring questions on G thereby factor through coloring questions on the individual treads.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

The structural study of such a minimal counterexample is the overarching motivation for the present line of work. This first paper establishes the foundational vocabulary — level sources, dual depth, tire graphs, and partial tire duals — on which subsequent papers in the series build. In particular, the companion paper [3] uses these definitions to develop nested-cycle structure theorems and chain-pigeonhole conjectures for tire annular subgraphs of G'.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is a set $S \subseteq V$ that is either

- a single vertex $\{v\}$ (a *vertex source*), or
- a set inducing a simple cycle in G — i.e. $G[S]$ is a simple cycle of length ≥ 3 (a *cycle source*).

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex. We write $L_d := \{v \in V : \ell_G(v) = d\}$ for the *level-d vertex set*, and abbreviate $L_{<d} := \bigcup_{d'<d} L_{d'}$ and $L_{\geq d} := \bigcup_{d' \geq d} L_{d'}$ (similarly $L_{>d}$, $L_{\leq d}$).

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Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \operatorname{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

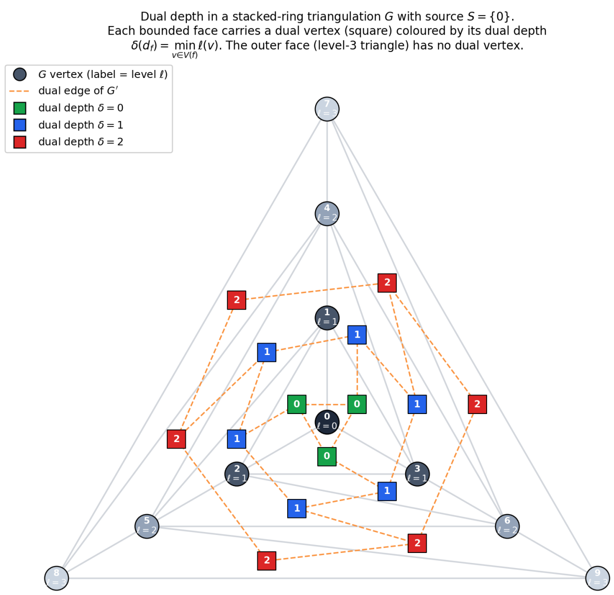

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.5 (Depth-d dual subgraph and its components)**.** For $d \geq 0$, the *depth-d dual subgraph* is

$$G'_d \;:=\; G'\big[\,\{d_f \in V(G') : \delta_G(d_f) = d\}\,\big],$$

the inner-dual subgraph induced on the dual vertices of dual depth d. For a connected component C' of G'_d we write

$$F_{C'} := \{f : d_f \in V(C')\}, \qquad V_{C'} := \bigcup_{f \in F_{C'}} V(f),$$

for its set of faces and the vertices of G bounding them, and $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$ for the closed planar region these faces cover.

Definition 1.6 (Tire graph)**.** A *tire graph* consists of a plane graph T together with an *outer boundary* $B_{\text{out}} \subseteq T$ and an *inner outerplanar graph* $O \subseteq T$ with $V(B_{\text{out}}) \cap V(O) = \emptyset$, where

- B_{out} is either a simple cycle of length ≥ 3 or a single vertex (a *degenerate outer boundary*);
- O is an outerplanar graph; its *inner boundary* B_{in} is the closed walk in O that traces the boundary of O's outer face in the inherited embedding, which is a simple cycle when O is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if $|V(O)| = 1$, we say T has a *degenerate inner boundary.*

At most one of $B_{\text{out}}, B_{\text{in}}$ may be degenerate. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \qquad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T, the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R. We call R the *tire tread* of T and write F_{ann} for this set of triangular faces (the *annular faces*).

When B_{out} is a simple cycle and O is 2-connected, the tread is a closed annulus. More generally, R is a closed planar region that may fail to be a 2-manifold at cut-vertices of O (where two "lobes" of the depth-d region meet at a single vertex); the inner boundary B_{in} is then a non-simple closed walk that visits the cut-vertex multiple times. The relaxed definition accommodates outerplanar inner graphs with bridges, cut-vertices, or multiple connected components. When either boundary is degenerate, the tread is a closed disk with that vertex as apex.

We summarize the data of a tire graph as the triple $T = (B_{\text{out}}, O, E_{\text{ann}})$, from which B_{in}, the annular faces F_{ann}, and the tread R are determined; we freely identify a tire graph with its underlying plane graph T.

Remark 1.7. Let $\mu = |V(B_{\text{out}})|$ and $\nu = |V(B_{\text{in}})|$. By Euler's formula on the tire tread R, the tire graph has $\mu + \nu$ triangular faces inside R and $|E_{\text{ann}}| = \mu + \nu$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(\mu, \nu) = 1$), there are $\mu + \nu - 1$ triangular faces and $|E_{\text{ann}}| = \mu + \nu - 1$.

Proposition 1.8 (Source-side simple-cycle property)**.** *Let G be a maximal planar graph with planar embedding Π_G and single-vertex source v_0. Let $d \geq 1$, $v \in L_d$, and let C' be a connected component of G'_d such that v is incident to some face in $F_{C'}$. Then the depth-d faces in $F_{C'}$ incident to v form a single contiguous arc in v's rotation in Π_G.*

Equivalently: for any such component, the source-side boundary of $R_{C'}$ is a simple cycle in L_d (no cut-vertices at level d).

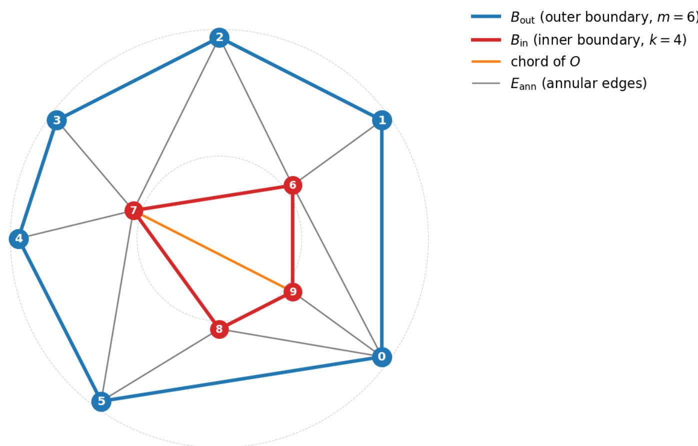

FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \ldots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \ldots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7\text{–}9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Proof. Suppose for contradiction that the depth-d faces in $F_{C'}$ at v lie in two or more disjoint arcs of v's rotation. Adjacent vertices in G differ in level by at most 1, so a face at v has depth exactly d iff both other vertices have level $\geq d$, and depth $\leq d-1$ iff at least one has level $d-1$. Hence the gaps between the depth-d arcs at v are populated by level-$(d-1)$ neighbours of v, occurring in at least two disjoint arcs of v's rotation. Pick p in one such gap and q in another.

The BFS ball $G[L_{<d}]$ is connected, so there exists a simple path P in $G[L_{<d}]$ from p to q. Define the closed walk

$$W \;:=\; v \to p \to P \to q \to v.$$

Every vertex of P lies in $L_{<d}$, while $\ell(v) = d$, so v is distinct from every vertex of P; P is simple, so its internal vertices are distinct; and $p \neq q$ since they lie in different gaps. Hence W is a simple cycle in G.

By the Jordan curve theorem, the planar embedding of W divides Π_G into two regions. In v's rotation, the edges $v - p$ and $v - q$ lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by W. By choice of p, q, the two arcs of depth-d faces at v in $F_{C'}$ lie in different regions of W (i.e., one arc on each side).

Since C' is connected in G' and contains depth-d faces in both arcs, there is a dual path $f_1, f_2, \ldots, f_k$ in G'_d with $f_1, f_k \in F_{C'}$ incident to v in different arcs, and with the intermediate faces $f_2, \ldots, f_{k-1}$ not incident to v (a shortest such dual path). Consecutive faces f_i, f_{i+1} share an edge e_i of G; for $i \geq 2$, both endpoints of e_i lie in $L_{\geq d}$ (since neither f_i nor f_{i+1} is incident to v, all six vertices of these two triangles lie in $L_{\geq d}$). In particular, e_i shares no endpoint with W except possibly v — and v is excluded from $f_2, \ldots, f_{k-1}$.

A planar edge with neither endpoint on a simple closed planar curve W has both of its incident faces on the same side of W. Applying this to each e_i ($i \geq 2$) inductively: starting from f_2 on the same side of W as f_1 (their shared edge $e_1 = w - w'$ opposite to v in f_1 has $w, w' \in L_{\geq d}$ and hence is not on W), the path $f_2 \to f_3 \to \cdots \to f_{k-1} \to f_k$ stays on one side of W.

But f_1 and f_k lie on different sides of W (by construction), contradicting the conclusion that the entire path lies on one side. □

Lemma 1.9 (Tire-component lemma)**.** *Let G be a maximal planar graph and let $S \subseteq V(G)$ be a level source. Fix a plane embedding Π_G of G in which S lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For $d \geq 0$, let C' be a connected component of the depth-d dual subgraph G'_d, with faces $F_{C'}$, bounding vertices $V_{C'}$, and region $R_{C'}$ as in Definition 1.5; let $T_{C'} := G[V_{C'}]$ inherit its embedding from Π_G.*

Then $T_{C'}$, with the inherited embedding, is a tire graph in the sense of Definition 1.6. Its outer boundary B_{out} is the side of $R_{C'}$ closer to S in Π_G, namely the level-d subgraph $G[V_{C'} \cap L_d]$ (a simple cycle or single vertex); its inner outerplanar graph is $O = G[V_{C'} \cap L_{d+1}]$, and its inner boundary B_{in} is the outer-face boundary closed walk of O in the inherited embedding (a simple cycle when O is 2-connected, a non-simple closed walk in general). The triangular faces of $T_{C'}$ inside the closed boundary region are exactly the faces of G in $F_{C'}$.

Proof. Outerplanarity of the two level parts. By construction S lies on the outer face of Π_G, so the outerplanarity lemma of [2] applies directly with (G, Π_G, S), giving that $G[L_{d'}]$ is outerplanar for each $d' \geq 0$. Subgraphs of outerplanar graphs are outerplanar, so $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are both outerplanar.

Layer containment. Each $f \in F_{C'}$ has at least one vertex at level d, and adjacent vertices in G differ in level by at most 1; combined with $\delta_G(d_f) = d$, this forces $V(f) \subseteq L_d \cup L_{d+1}$. Hence $V_{C'} \subseteq L_d \cup L_{d+1}$, and $T_{C'}$ has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$.

Boundary edges are monochromatic in level. Each edge e on $\partial R_{C'}$ separates a face $f \in F_{C'}$ from a face $f' \notin F_{C'}$. Because f and f' share the edge e, their dual vertices are adjacent in G'; if both had depth d they would lie in the same component of G'_d, contradicting $d_f \in C'$ and $d_{f'} \notin C'$. Hence $\delta_G(d_{f'}) \neq d$; combined with the bounded-step property of δ across G'-adjacent faces, $\delta_G(d_{f'}) \in \{d-1, d+1\}$.

- If $\delta_G(d_{f'}) = d-1$, the third vertex w of $f' = \{u, v, w\}$ (where u, v are the endpoints of e) has $\ell(w) = d-1$. Each of u, v has $\ell \in \{d, d+1\}$ (from $V(f) \subseteq L_d \cup L_{d+1}$) and is adjacent to w, forcing $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$.
- If $\delta_G(d_{f'}) = d+1$, then all three vertices of f' lie in $L_{\geq d+1}$, so in particular $\ell(u) = \ell(v) = d+1$.

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of $\partial R_{C'}$ is monochromatic in level.

Boundary structure. Each connected component of $\partial R_{C'}$ traces a closed walk in G that, by the monochromaticity above, lies entirely in L_d or entirely in L_{d+1}. By Proposition 1.8, the depth-d faces of $F_{C'}$ at any $v \in L_d \cap V_{C'}$ form a single contiguous arc in v's rotation, so the source-side boundary walk visits each L_d-vertex of $V_{C'}$ exactly once: it is a simple cycle. At vertices $v \in L_{d+1} \cap V_{C'}$ the depth-d faces may split into multiple arcs of v's rotation; this corresponds exactly to v being a cut-vertex of O, and the inner-side boundary walk visits v correspondingly many times — which is already accommodated by Definition 1.6 (where B_{in} is the outer-face boundary closed walk of O, not necessarily a simple cycle).

Outer boundary. Because S lies on the outer face of Π_G, the boundary curve(s) of $R_{C'}$ on the L_d side are closer to S in the embedding. In the inherited embedding of $T_{C'}$, the unique unbounded face is the merged region containing the rest of Π_G outside $R_{C'}$ on the S side, so its boundary — a simple cycle on L_d (or a single vertex when $V_{C'} \cap L_d = \{v_0\}$, the $d = 0$ case) — serves as B_{out}. We set $B_{\text{out}} := G[V_{C'} \cap L_d]$ if this is a cycle, and the single vertex $\{v_0\}$ in the degenerate case.

Inner outerplanar graph. By the outerplanarity lemma of [2], $G[V_{C'} \cap L_{d+1}]$ is outerplanar. We set $O := G[V_{C'} \cap L_{d+1}]$. The boundary curve(s) of $R_{C'}$ on the L_{d+1} side are exactly the boundary of O's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around O from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary B_{in}. No further restriction on O's internal structure is needed: when $R_{C'}$ has more than two boundary components in the surface-classification sense (i.e. several disjoint simple cycles on L_{d+1}), these correspond precisely to the multiple connected components or bridge crossings of O, and the outer-face boundary closed walk of O captures them collectively.

Tire structure. The triangular faces of $T_{C'}$ inside the closed boundary region are by construction the depth-d faces in $F_{C'}$, and the edges of $T_{C'}$ are $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$ where E_{ann} are the edges of G between $V_{C'} \cap L_d$ and $V_{C'} \cap L_{d+1}$ that bound a face of $F_{C'}$. □

Theorem 1.10 (Tire treads partition the bounded faces)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a level source lying on the outer face. For each $d \geq 0$ and each connected component C' of G'_d, let $T^{(d,C')}$ denote the tire graph supplied by Lemma 1.9, with tire tread $R_{C'} \subseteq |\Pi_G|$. Then the collection of treads*

$$\mathcal{R}(G,S) \;:=\; \left\{ R_{C'} \,:\, d \geq 0,\ C' \text{ a connected component of } G'_d \right\}$$

partitions the bounded part of $|\Pi_G|$:

(i) *every bounded face f of G is contained in exactly one tread $R_{C'} \in \mathcal{R}(G,S)$;*
(ii) *distinct treads in $\mathcal{R}(G,S)$ have disjoint interiors and may share only boundary edges or vertices.*

Proof. Existence and uniqueness. Each bounded face $f \in F(G)$ has a uniquely-defined dual depth $\delta_G(d_f) \in \mathbb{Z}_{\geq 0}$, so the dual vertex d_f lies in G'_d for $d = \delta_G(d_f)$ and in no other $G'_{d'}$. Within G'_d, the vertex d_f belongs to exactly one connected component C'. By Lemma 1.9, $F_{C'}$ is precisely the set of faces $f' \in F(G)$ with $d_{f'} \in V(C')$; in particular $f \in F_{C'}$, hence $f \subseteq R_{C'}$.

For any other tread $R_{C''} \in \mathcal{R}(G,S)$, the component C'' is either at a different depth $d' \neq d$ (in which case $F_{C''}$ consists of depth-d' faces and $f \notin F_{C''}$) or at depth d but a different component $C'' \neq C'$ (in which case the two components are vertex-disjoint in G'_d, so again $f \notin F_{C''}$). In both cases $f \notin R_{C''}$ (more precisely, f is not one of the triangular faces of G in $F_{C''}$, so f's interior is not contained in $R_{C''}$).

Disjoint interiors. Each tread $R_{C'}$ is the union of its triangular faces $F_{C'} \subseteq F(G)$; distinct treads correspond to disjoint $F_{C'}$ (by the argument above), and the interiors of distinct G-faces are disjoint. Hence interiors of distinct treads are disjoint.

Coverage. Conversely, every bounded $f \in F(G)$ has $d_f \in V(G')$ with some dual depth d, and thus lies in $R_{C'}$ where C' is its component of G'_d. So $\bigcup_{R\in\mathcal{R}(G,S)} R$ contains every bounded face of G. □

Remark 1.11. Either boundary part of $T_{C'}$ in Lemma 1.9 may be degenerate. At $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ as the degenerate *outer* boundary and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G) as the inner boundary. Symmetrically, at $d = D_{\max}$, $V_{C'} \cap L_{D_{\max}+1} = \emptyset$ degenerates to a single deepest vertex serving as the *inner* boundary, with the level-$D_{\max}$ cycle as the outer boundary.

Remark 1.12. Two structural features of $R_{C'}$ that might at first appear to obstruct the tire-graph conclusion are both already accommodated by Definition 1.6:

Cut-vertices of O. A vertex $v \in V_{C'} \cap L_{d+1}$ may have the faces of $F_{C'}$ incident to it split into two or more arcs in v's rotation in Π_G, separated by faces of higher depth. This corresponds exactly to v being a cut-vertex of $O = G[V_{C'} \cap L_{d+1}]$, and the inner boundary closed walk B_{in} then visits v multiple times — once for each arc. No additional hypothesis is needed.

Multi-hole topology of $R_{C'}$. Even when $R_{C'}$ encloses several disjoint depth-$> d$ sub-regions, the inner outerplanar graph O captures the multi-hole structure as a disconnected or non-2-connected outerplanar graph (with bridges or multiple components), and its outer-face boundary closed walk serves as B_{in} traversing bridges twice and visiting cut-vertices multiple times.

In the special case $d = 0$ with single-vertex source $S = \{v_0\}$, $R_{C'}$ is the star of v_0, a topological closed disk with one boundary cycle (the link of v_0); the corresponding tire graph has degenerate outer boundary $\{v_0\}$.

Theorem 1.13 (Inner dual of a tire tread is outerplanar). *Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph, and let Γ be the graph on vertex set $\{d_f : f \in F_{\text{ann}}\}$ with an edge $d_f d_{f'}$ for each interior annular edge of T (= each edge of T whose two incident faces both lie in F_{ann}). Equivalently, Γ is the subgraph induced on F_{ann} of the* full tire dual $D(T)$ *— the dual of T taken over all of its triangular faces, in which each boundary edge of R contributes a degree-*1 *vertex. Then Γ is outerplanar.*

Moreover, Γ admits a planar embedding as a (possibly non-simple) Hamilton walk through every d_f, plus zero or more non-crossing chords.

Proof. We argue by cases on whether the tire tread R is a disk or an annulus.

Case 1: R is a closed disk (one of $B_{\text{out}}, B_{\text{in}}$ degenerate, by Definition 1.6). Let v_0 be the degenerate-boundary vertex (the apex) and let $k = |B_{\text{non-deg}}|$ be the length of the non-degenerate boundary cycle. The triangulation of R is a *fan* of k triangles around v_0: each triangle has the form $\{v_0, u_i, u_{i+1}\}$ where $u_1, \ldots, u_k$ are the boundary-cycle vertices in cyclic order. Each triangle has two spoke edges (= the two edges incident to v_0, shared with the two neighbouring fan triangles) and one boundary edge (in $B_{\text{non-deg}}$, contributing a leaf in $D(T)$ but no edge in Γ). Hence every d_f has Γ-degree exactly 2, and Γ is a single cycle of length k. Cycles are outerplanar.

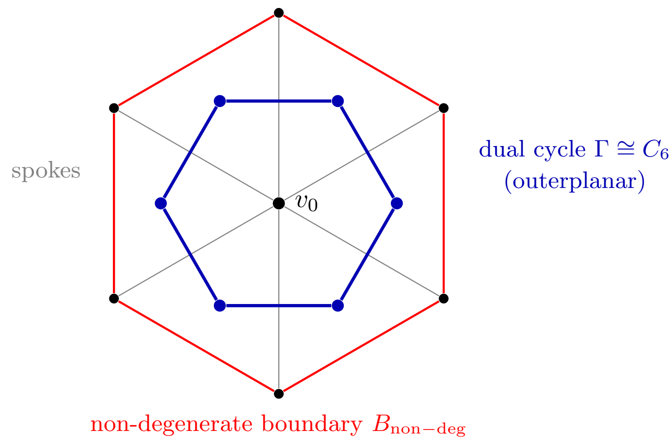

FIGURE 3. Case 1 (R = disk, $k = 6$). The apex v_0 sits at the centre; the non-degenerate boundary $B_{\mathrm{non-deg}}$ (red) is the hexagonal outer cycle; spokes (grey) triangulate the disk into a fan of 6 triangles around v_0. Each triangle has two spoke edges (interior, contributing Γ-edges) and one boundary edge (contributing a leaf in $D(T)$, no Γ-edge). The inner dual Γ (blue) is the cycle C_6 formed by the six annular face centroids, a manifestly outerplanar graph.

See Figure 3 for the disk case ($k = 6$).

Case 2: R is an annulus (both B_{out} and B_{in} non-degenerate). We construct an explicit outerplanar embedding of Γ as a Hamilton walk plus non-crossing chords.

Step 1: Cyclic ordering of F_{ann}. The boundary of the annular tread is the disjoint union $\partial R = B_{\mathrm{out}} \sqcup \overline{B_{\mathrm{in}}}$ (viewing B_{in} as a closed walk traced in the appropriate orientation). Each boundary edge of R is incident to exactly one annular face: walking around B_{out} in cyclic order produces a sequence $f_1^{\mathrm{out}}, f_2^{\mathrm{out}}, \dots, f_\mu^{\mathrm{out}}$ of (not necessarily distinct) annular faces, one per B_{out}-edge; similarly walking around B_{in} produces a sequence $f_1^{\mathrm{in}}, \dots, f_{\nu_\partial}^{\mathrm{in}}$ where ν_∂ is the length of the inner-boundary walk. Pick any spoke $e^\star = uw \in E_{\mathrm{ann}}$ with $u \in V(B_{\mathrm{out}})$ and $w \in V(B_{\mathrm{in}})$; cut R along $e^\star$. This converts the annulus into a closed disk $\tilde{R}$ whose boundary walks once around B_{out}, once along $e^\star$, once around B_{in} in reverse, and once back along $e^\star$. Concatenating the two boundary sequences (in the order dictated by this disk traversal) yields a single cyclic sequence

$$\mathcal{S} = (f_1^{\mathrm{out}}, \dots, f_\mu^{\mathrm{out}}, f_1^{\mathrm{in}}, \dots, f_{\nu_\partial}^{\mathrm{in}})$$

of annular faces with multiplicities.

Step 2: The Hamilton walk. Consecutive entries of $\mathcal{S}$ correspond either to the same annular face (when two adjacent boundary edges meet at a vertex incident to a single annular face) or to two annular faces sharing an interior edge of E_{ann}. In the former case the walk stays at one Γ-vertex; in the latter it uses one Γ-edge. The resulting closed walk in Γ visits every face that appears in $\mathcal{S}$ at least once.

If every $f \in F_{\mathrm{ann}}$ appears in $\mathcal{S}$ (i.e. every annular face has at least one boundary edge of R), the walk is a Hamilton walk in Γ, and we are done up to Step 3. Each annular face with two boundary edges contributes a vertex visited twice; each with three contributes a vertex visited three times.

If some $f \in F_{\mathrm{ann}}$ does not appear in $\mathcal{S}$ (i.e. has no boundary edge of R), then all three edges of f are interior annular edges, so d_f has degree 3 in Γ. Such a face is "trapped" in the interior of the dual graph and appears as the endpoint of a chord. Extend the walk by: whenever it crosses an interior annular edge e shared with a boundary-free face f, detour through f and back. After finitely many such detours (one per boundary-free face), the walk becomes a Hamilton walk visiting every d_f.

Step 3: Non-crossing chords. The Γ-edges not used by the Hamilton walk constructed in Step 2 are the remaining interior annular edges. Each such edge $e \in E_{\mathrm{ann}}$ corresponds to a chord between two non-adjacent positions of $\mathcal{S}$. In the inherited planar embedding of Γ in R, these chords are drawn as straight segments between annular triangle centroids; *they do not cross* because the underlying E_{ann} edges they cross are themselves non-crossing in the planar embedding of T.

Step 4: Outerplanar embedding. We now lay out Γ as follows: place the $|F_{\mathrm{ann}}|$ vertices on a circle in the cyclic order given by $\mathcal{S}$ (treating multiply-visited faces as single circle vertices). Connect consecutive vertices on the circle by the Hamilton-walk edges, which forms the closed walk. Draw the remaining edges as chords inside the circle. Because the chords were non-crossing in T's planar embedding, they remain non-crossing here. All vertices lie on the outer face (the unbounded region outside the circle), making Γ outerplanar. □ □

Remark 1.14. In the *spoke-only* case (Definition 1.6 with O 2-connected and E_{ann} consisting only of spokes), every annular face has exactly one boundary edge, every d_f has Γ-degree 2, and the construction of the Theorem 1.13 proof reduces to the classical Hamilton cycle $\Gamma \cong C_{\mu+\nu}$ with zero chords.

Remark 1.15. When O has a bridge $e_{\mathrm{br}} \in E(O)$ whose two incident faces are annular triangles, e_{br} contributes an interior annular edge in Γ rather than two leaves in $D(T)$ (see Definition 1.7 of [3]). The two bridge-incident annular triangles have Γ-degree 3; the resulting Γ has the structure of a Hamilton cycle of length $\mu + \nu_\partial$ plus a single chord (length 1). This corresponds to the theta graph $\Theta(1, b, c)$ identified empirically in [3], which has no $K_{2,3}$ subdivision (since one of the three paths has length 1 and so contributes no degree-2 branch vertex), hence is outerplanar as predicted.

Theorem 1.16 (Tait correspondence: 4-colorings of a tire vs 3-edge-colorings of its inner dual)**.** *Let $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ be a tire graph (viewed as an annular triangulation of its tire tread R) and let Γ be its inner dual (Theorem 1.13). Then*

$$\#\big\{\textit{proper 4-vertex-colorings of } T\big\}/|S_4| \;=\; \#\big\{\textit{proper 3-edge-colorings of } \Gamma\big\}/|S_3|.$$

That is, the number of 4-vertex-colorings of T up to permutation of the colour set $\{0,1,2,3\}$ equals the number of 3-edge-colorings of Γ up to permutation of the colour set $\{1,2,3\}$.

Proof. The argument is the classical Tait correspondence [1] adapted to the annular triangulation T. Encode the four colours of a proper 4-vertex-coloring $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. For each interior annular edge e of T (whose two incident faces both lie in F_{ann}, contributing a Γ-edge e^*), set

$$\chi^*(e^*) \;:=\; c(u) + c(v) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \text{where } u, v \text{ are the endpoints of } e.$$

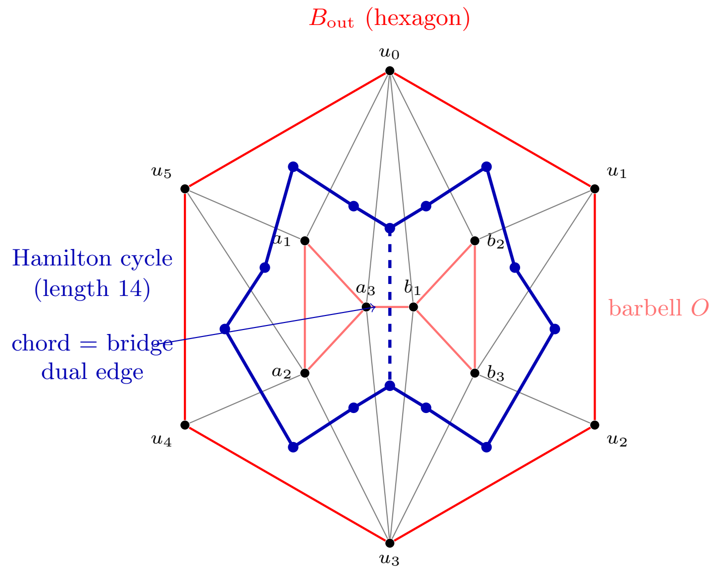

FIGURE 4. Case 2 (R = annulus) with O a barbell. B_{out} is the outer hexagon (red); O has two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ joined by the bridge a_3–b_1 (all light red). The annulus is triangulated by 14 annular triangles: 6 "outer-cap" triangles (one per outer edge), 6 "inner-cap" triangles (one per non-bridge edge of O), and 2 "bridge-cap" triangles $\{u_0, a_3, b_1\}$ and $\{u_3, a_3, b_1\}$ adjacent to the bridge. Each blue dot sits at the centroid of an annular triangle; blue edges connect dual vertices whose triangles share an interior annular edge (spoke or bridge). The two bridge-cap vertices have Γ-degree 3 (their triangles have no boundary edge) and are joined by the dashed blue *chord* corresponding to the bridge; the remaining 13 edges form the Hamilton cycle that wraps around the annulus. All 14 vertices lie on the outer face of the cycle-with-chord embedding, so $\Gamma \cong \Theta(1, 7, 7)$ is outerplanar.

Since $c(u) \neq c(v)$, we have $\chi^*(e^*) \neq 00$, so χ^* takes values in $\{01, 10, 11\}$, which we identify with the 3-edge-coloring palette $\{1, 2, 3\}$.

Properness. At each Γ-vertex d_f corresponding to an annular triangle $f = \{u, v, w\}$, the three incident Γ-edges (one per cycle-edge of f) carry colours $c(u) + c(v)$, $c(v) + c(w)$, $c(u) + c(w)$. These three elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ sum to 0 and are pairwise distinct (their pairwise differences are $c(u) - c(w)$, $c(v) - c(u)$, $c(w) - c(v)$, each nonzero), so they form a permutation of $\{01, 10, 11\}$ — a proper edge colouring at d_f.

Surjectivity onto cosets. Given a proper 3-edge-coloring χ^* of Γ, the equation $c(u) + c(v) = \chi^*(e^*)$ admits exactly $|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$ solutions $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ (a global translation is the only freedom). Hence the map $c \mapsto \chi^*$ is 4-to-1.

Count. Therefore $\#\{4\text{-colorings of } T\} = 4 \cdot \#\{3\text{-edge-colorings of } \Gamma\}$. Dividing by $|S_4| = 24$ on the left and $|S_3| = 6$ on the right (since S_4 acts faithfully on the 4-colorings and S_3 on the 3-edge-colorings, and the 4-to-1 map respects the $S_4/S_3 \cong S_3$ quotient via the natural surjection $S_4 \twoheadrightarrow S_3$) gives the stated equality. □

Remark 1.17. Theorem 1.16 reduces the 4-colouring count of a tire to the 3-edge-coloring count of its outerplanar inner dual Γ. For the cycle case $\Gamma \cong C_{\mu+\nu}$ (the spoke-only case of Remark 1.14), the cycle chromatic polynomial at 3 colours gives $2^{\mu+\nu}+2(-1)^{\mu+\nu}$. For an inner dual with one or more non-crossing chords, the count depends on the chord structure, not just on the pair (number of vertices, number of chords): two outerplanar graphs with the same number of vertices and number of chords can have different proper 3-edge-coloring counts depending on how the chords are arranged (nested, sequential, sharing vertices, etc.). Every such count can nevertheless be computed in linear time by tree-decomposition methods, since outerplanar graphs have treewidth at most 2 and the edge-chromatic polynomial admits a deletion–contraction recursion that respects the cycle-plus-chord structure.

Theorem 1.18 (Tire treads form a rooted tree under face containment)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a single-vertex level source $\{v_0\}$ lying on the outer face of Π_G. The collection $\mathcal{R}(G,S)$ of tire treads (Theorem 1.10) carries a canonical rooted tree structure $\mathcal{T}(G,S)$ defined as follows.*

- Root. *The depth-*0 *tire tread T_0 — the unique tire produced by Lemma 1.9 at $d = 0$, with degenerate outer boundary $B_{\text{out}} = \{v_0\}$ and inner outerplanar graph $O^{(T_0)} = G[L_1]$ — is the root.*
- Parent. *For each tire tread T_c at depth $d \geq 1$, its outer boundary $B^{(T_c)}_{\text{out}}$ is a cycle in L_d. The* parent *of T_c is the unique tire tread T_p at depth $d-1$ whose inner outerplanar graph $O^{(T_p)}$ has $B^{(T_c)}_{\text{out}}$ as the boundary cycle of one of its bounded faces. Equivalently, R_c lies inside this bounded face of $O^{(T_p)}$ (which is itself the region of the plane cut off by $B^{(T_c)}_{\text{out}}$ on the side away from S).*
- Children. *The children of a tire tread T_p are in bijection with those bounded faces of $O^{(T_p)}$ whose interiors contain at least one vertex of G at level $\geq d+2$ — equivalently, with the connected components of G'_{d+1} whose tires have outer boundary cycle equal to a bounded face of $O^{(T_p)}$.*

Every tire tread except T_0 has exactly one parent; a tire tread may have zero, one, or several children.

Proof. Root is well-defined. At $d = 0$ with single-vertex source $S = \{v_0\}$, the dual subgraph G'_0 is connected (every face of G incident to v_0 has dual depth 0, and they form a single fan around v_0). By Lemma 1.9, the unique component of G'_0 gives the depth-0 tire T_0 described above.

Existence of parent. Fix a tire tread T_c at depth $d \geq 1$ arising from a connected component C'_c of G'_d. Its outer boundary $B^{(T_c)}_{\text{out}} = G[V_{C'_c} \cap L_d]$ is a simple cycle in L_d (Lemma 1.9; the source-side boundary of a tire is always a simple cycle, by Proposition 1.8). The faces of G immediately outside $B^{(T_c)}_{\text{out}}$ on the side facing S have depth $d-1$ (one of their three vertices lies in L_{d-1}, two in L_d). Let C'_p be the connected component of G'_{d-1} containing the dual vertex of any such face.

Uniqueness of parent. $B_{\text{out}}^{(T_c)}$ is a single simple cycle in G, with a well-defined "S-side" (the side of the cycle closer to v_0 in Π_G). The depth-$(d-1)$ faces lying on this side form a single contiguous arc around $B_{\text{out}}^{(T_c)}$ in the dual — they are all G'-adjacent in sequence (each pair of consecutive arc faces shares an edge in $B_{\text{out}}^{(T_c)}$). Hence they all lie in the same connected component C'_p of G'_{d-1}, which is therefore unique.

$B_{\text{out}}^{(T_c)}$ *bounds a face of* $O^{(T_p)}$. The parent tire T_p has $V(O^{(T_p)}) = V_{C'_p} \cap L_d \supseteq V(B_{\text{out}}^{(T_c)})$. The cycle $B_{\text{out}}^{(T_c)}$ is a subgraph of $O^{(T_p)}$ that bounds a face of $O^{(T_p)}$ in the inherited embedding: the cycle traces around a depth-$\geq d+1$ region (containing R_c and any descendants of T_c), which is exactly a bounded face of $O^{(T_p)}$.

Children description. The bounded faces of $O^{(T_p)}$ are in bijection with the connected components of G'_d whose faces lie inside those bounded regions (= one component per bounded face, by an argument analogous to the existence-and- uniqueness step above, applied one level deeper).

Tree property. Every non-root T_c has a unique parent at strictly smaller depth. Iterating the parent map strictly decreases depth, terminating at T_0. No cycles can form (depth is monotone). Hence $\mathcal{T}(G, S)$ is a rooted tree. $\square$

Remark 1.19. A parent tire T_p has multiple children precisely when its inner outerplanar graph $O^{(T_p)}$ has multiple bounded faces with non-trivial interiors (= containing depth-$\geq d+2$ vertices of G). This happens, for instance, when $O^{(T_p)}$ has chords or cut-vertices that subdivide its inner region, or when $O^{(T_p)}$ has multiple connected components in $G[L_{d+1}] \cap V_{C'_p}$. By contrast, if $O^{(T_p)}$ is a simple cycle (the spoke-only case of Remark 1.14) with a non-empty interior, T_p has exactly one child.

Theorem 1.20 (Tire-tree decomposition)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $v_0 \in V(G)$. The tree of tire treads $\mathcal{T}(G, \{v_0\})$ of Theorem 1.18* decomposes G into nested tires*: it is a finite rooted tree, rooted at the depth-*0 *tread containing v_0, whose nodes (tire treads) partition the bounded faces of G (Theorem 1.10).*

This decomposition is moreover self-similar*. For any tread T in $\mathcal{T}(G, \{v_0\})$ at depth $d \geq 1$, with outer-boundary cycle $C_T := B_{\text{out}}^{(T)}$, let G_T be the sub-graph of G induced by C_T together with all vertices of G lying in the closed planar region $R_T \subset |\Pi_G|$ bounded by C_T on the side of C_T away from v_0. Then:*

- (D1) *G_T, with the embedding inherited from Π_G, is a* triangulated disk*: every bounded face is a triangle, and the outer face is bounded by C_T.*
- (D2) *Taking C_T as a cycle source of G_T (so C_T has level* 0 *in G_T and the BFS-from-C_T levels in G_T equal $\ell_G(\cdot) - d$ on $V(G_T)$), the construction of Theorem 1.18 extends to give a rooted tree of tire treads $\mathcal{T}(G_T, C_T)$ whose depth-*0 *root tread has $B_{\text{out}} = C_T$ and inner outerplanar graph $O = O^{(T)}$.*
- (D3) *$\mathcal{T}(G_T, C_T)$ is canonically iso to the sub-tree of $\mathcal{T}(G, \{v_0\})$ rooted at T, preserving outer-boundary cycles, inner outerplanar graphs, and the parent–child face correspondence.*

In short: pick any vertex $v_0 \in V(G)$ to root the global tree $\mathcal{T}(G, \{v_0\})$ describing the whole graph; pick any tread T in this tree; then T is itself the root of a local tree $\mathcal{T}(G_T, C_T)$ describing the triangulated disk of G inside C_T, with C_T as cycle source. Maximal planar graphs decompose into nested trees of tire treads.

Proof. Decomposition. Theorem 1.18 gives the rooted tree structure of $\mathcal{T}(G, \{v_0\})$, with root the depth-0 tread containing v_0; Theorem 1.10 gives that its tire treads partition the bounded faces of G. Finiteness of the tree is immediate from finiteness of G.

(D1) G_T is a triangulated disk. By Lemma 1.9 applied to the component of G'_d that gives rise to T, the outer boundary $C_T = B^{(T)}_{\text{out}}$ is a simple cycle in L^G_d. By the Jordan curve theorem, C_T separates $|\Pi_G| \setminus C_T$ into two open regions; R_T is the closure of the one not containing v_0. The bounded faces of G_T in its inherited embedding are exactly the bounded faces of G contained in R_T, each of which is a triangle since G is a triangulation. The unbounded face of G_T's embedding is the complement of R_T, whose boundary is C_T.

(D2) Level shift. We show $\operatorname{dist}_{G_T}(v, C_T) = \ell_G(v) - d$ for every $v \in V(G_T)$. When $v \in C_T$ both sides equal 0, so fix $v \in V(G_T) \setminus C_T$.

Step 1: $\operatorname{dist}_G(v, C_T) = \ell_G(v) - d$. A shortest G-path from v to v_0 must visit C_T, since v and v_0 lie in different open regions of $|\Pi_G| \setminus C_T$; let w be its first C_T-vertex. The v-to-w sub-path has length $\geq \operatorname{dist}_G(v, C_T)$ and the w-to-v_0 sub-path has length $\ell_G(w) = d$, so $\ell_G(v) \geq \operatorname{dist}_G(v, C_T) + d$. Conversely, concatenating a shortest G-path from v to a nearest C_T-vertex w' with a shortest G-path from w' to v_0 gives a v-to-v_0 path of length $\operatorname{dist}_G(v, C_T) + d$, so $\ell_G(v) \leq \operatorname{dist}_G(v, C_T) + d$.

Step 2: $\operatorname{dist}_{G_T}(v, C_T) = \operatorname{dist}_G(v, C_T)$. The inequality $\geq$ is automatic since $G_T \subseteq G$. For $\leq$, pick a shortest G-path π from v to C_T; we may assume π has no internal vertex in C_T (truncate otherwise). Any internal vertex of π then lies in the same open region of $|\Pi_G| \setminus C_T$ as v, i.e. in $R_T \setminus C_T \subseteq V(G_T)$; every edge of π has both endpoints in $V(G_T)$ and so lies in $E(G_T)$. Hence π is a path in G_T realising $\operatorname{dist}_G(v, C_T)$.

Combining the two steps yields $\operatorname{dist}_{G_T}(v, C_T) = \ell_G(v) - d$, as claimed.

(D3) Tree iso. By (D2), $L^{G_T}_k = L^G_{d+k} \cap V(G_T)$ for every $k \geq 0$. For a bounded face f of G_T, dual depth in G_T equals $\min_{u \in V(f)} \ell_{G_T}(u) = \min_{u \in V(f)} \ell_G(u) - d = \delta_G(d_f) - d$. Hence the inner-dual subgraph $(G_T)'_k$ at depth k in G_T is the induced subgraph of G'_{d+k} on the faces of G lying in R_T, and two such faces are dual-adjacent in G'_T iff they are dual-adjacent in G' (the shared edge is in $E(G_T)$).

Step 3: components of $(G_T)'_k$ are precisely the depth-$(d+k)$ descendants of T in $\mathcal{T}(G, \{v_0\})$. We show by induction on k that a component C' of G'_{d+k} has $F_{C'} \subseteq R_T$ iff C' is a depth-$(d+k)$ descendant of T.

For $k = 0$: the components of G'_d are the depth-d treads; the component giving rise to T has its faces in T's tread region $R \subseteq R_T$, while any other depth-d tread T'' has $C_{T''}$ disjoint from C_T and lying in a different bounded face of $O^{(T''_p)}$ at depth $d-1$, hence $R_{T''} \cap R_T = \emptyset$.

For $k \geq 1$: by Theorem 1.18, each component C' of G'_{d+k} has a unique parent C'_p at depth $d+k-1$, with $B^{(C')}_{\text{out}}$ bounding a face of $O^{(C'_p)}$; equivalently $R_{C'}$ lies inside that bounded face, hence inside $R_{C'_p}$. By the induction hypothesis $R_{C'_p} \subseteq R_T$ iff C'_p is a descendant of T at depth $d+k-1$, and $R_{C'} \subseteq R_{C'_p}$, so $R_{C'} \subseteq R_T$ iff C' is a descendant of T at depth $d+k$.

Step 4: tread data and child–face correspondence. The Tire-component lemma (Lemma 1.9) and the source-side simple-cycle property (Proposition 1.8) extend verbatim to the cycle-sourced triangulated disk (G_T, C_T): the proofs use only the triangular structure of bounded faces, the local arrangement of faces around each vertex's rotation, and the connectivity of the BFS ball $G_T[L_{<k}^{G_T}]$ (which holds for every $k \geq 1$ since $L_0^{G_T} = V(C_T)$ is connected as a cycle and each higher level is BFS-adjacent to the previous). Applied to each component of $(G_T)'_k$, the lemma produces a tire graph with outer boundary B_{out}, inner outerplanar graph O, and tread region R identical to those produced by the corresponding component of G'_{d+k} in $\mathcal{T}(G, \{v_0\})$, since these data depend only on level-$d+k$ and level-$(d+k+1)$ vertices and the bounded faces in between — all of which are unchanged when restricting to G_T.

The depth-0 case ($k = 0$) gives a single component, namely the one producing T, with root tread $B_{\text{out}} = C_T$ and $O = O^{(T)}$.

The parent–child face correspondence of Theorem 1.18 is preserved: for any tread T' in $\mathcal{T}(G_T, C_T)$ at depth k, its children correspond to non-trivial bounded faces of $O^{(T')}$, and the bounded faces of $O^{(T')}$ together with the descendant-side interior of each are identical in G_T and in G.

Combining Steps 3 and 4: the bijection $C' \leftrightarrow C'$ (component of $(G_T)'_k$ to corresponding component of G'_{d+k} inside R_T) lifts to a rooted-tree iso $\mathcal{T}(G_T, C_T) \to$ sub-tree of $\mathcal{T}(G, \{v_0\})$ rooted at T, preserving outer boundaries, inner outerplanar graphs, and the parent–child face correspondence. $\square$

Remark 1.21. Combining Theorem 1.10 (treads partition the bounded faces of G) with Theorem 1.18 (treads form a rooted tree), any proper coloring problem on G's bounded faces factors through:

- local coloring problems on each tread (the inner dual of each tread is outerplanar by Theorem 1.13), plus
- consistency constraints along parent-child interfaces (the cycle $B_{\text{out}}^{(T_c)}$ shared between a child and the face of its parent's $O^{(T_p)}$).

This is the structural setup underlying the chain-pigeonhole program for tire treads.

Definition 1.22 (Seam)**.** A *seam* of a maximal planar graph G is a simple cycle $C \subset G$ such that, for some vertex $v_0 \in V(G)$, $C = B_{\text{out}}^{(T)}$ for some non-root tread T in $\mathcal{T}(G, \{v_0\})$.

By Theorem 1.20, every seam C separates G into:

- the *seam interior* G_T, the triangulated disk on the T-descendant side of C;
- the *seam exterior* $G_C^{\text{ext}} := G \setminus \text{int}(G_T)$, the triangulated polygon with outer face bounded by C on the side containing v_0;

both sharing C. A seam is *non-trivial* if both $V(G_T) \setminus V(C)$ and $V(G_C^{\text{ext}}) \setminus V(C)$ are non-empty.

For any seam C and either side $X \in \{G_T, G_C^{\text{ext}}\}$, write

$$\text{Col}(X \mid C) \;:=\; \big\{\, c|_{V(C)} \;:\; c \text{ a proper 4-colouring of } X \,\big\} \;\subseteq\; \{1,2,3,4\}^{V(C)}$$

for the set of C-restricted 4-colourings induced by 4-colourings of X (each element is a proper 4-colouring of the cycle C).

Definition 1.23 (Partial tire tree)**.** Let T_r be a tire tread in $\mathcal{T}(G, S)$ with outer boundary cycle $C_{T_r} = B_{\text{out}}^{(T_r)}$, and let G_{T_r} be the triangulated disk inside C_{T_r} given by Theorem 1.20. The *partial tire tree* with root T_r, written $G_{T_r}^{\circ}$, is the induced subgraph of G on the vertex set $V(G_{T_r}) \setminus V(C_{T_r})$ — i.e. G_{T_r} with the seam-cycle vertices removed.

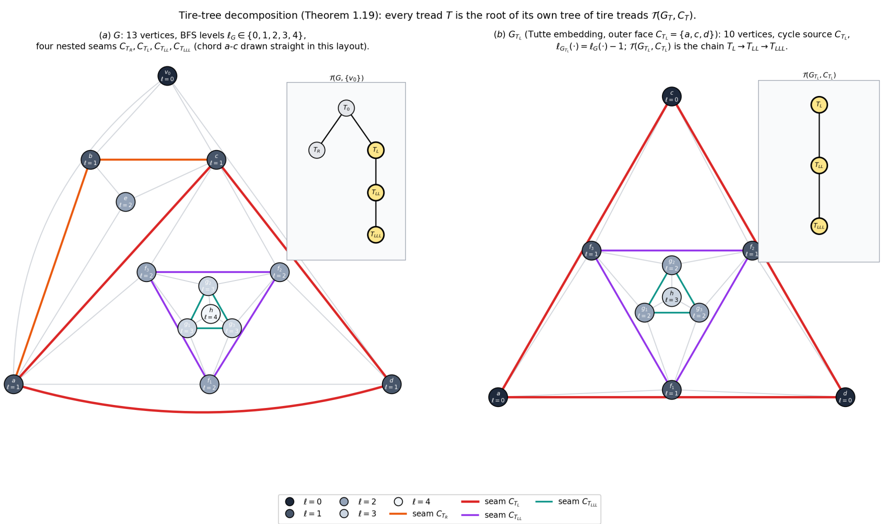

FIGURE 5. Tire-tree decomposition (Theorem 1.20) on a 13-vertex maximal planar example G with five BFS levels. (a) G with vertex source v_0 and $\ell_G \in \{0,1,2,3,4\}$; four nested seams are highlighted, $C_{T_R} = \{a,b,c\}$ (orange), $C_{T_L} = \{a,c,d\}$ (red, including the chord a-c shared with C_{T_R}), $C_{T_{LL}} = \{f_1, f_2, f_3\}$ (purple), $C_{T_{LLL}} = \{g_1, g_2, g_3\}$ (teal). Inset: the rooted tree of tire treads $\mathcal{T}(G, \{v_0\})$ branches at T_0 into the leaf T_R (containing e) and a chain $T_L \to T_{LL} \to T_{LLL}$ (the highlighted sub-tree). (b) The disk G_{T_L} inside the seam C_{T_L}, drawn standalone with C_{T_L} as cycle source and vertex labels rotated to match the new (cycle-source) role of the boundary triangle. $\ell_{G_{T_L}}(\cdot) = \ell_G(\cdot) - 1$ on $V(G_{T_L})$ (verified by the generator script), and $\mathcal{T}(G_{T_L}, C_{T_L})$ is the chain $T_L \to T_{LL} \to T_{LLL}$, iso to the highlighted sub-tree of (a).

Equivalently, $V(G^\circ_{T_r})$ is the set of vertices of G strictly inside C_{T_r} on the side away from the level source, and $E(G^\circ_{T_r})$ consists of the edges of G both of whose endpoints lie in this strict interior. The tree-of-tire-treads structure of $G^\circ_{T_r}$ is the sub-tree of $\mathcal{T}(G, S)$ rooted at T_r, with T_r's outer boundary peeled away.

Lemma 1.24 (Seam edges are shared by at most one other depth-d seam)**.** *Let G be a maximal planar graph with single-vertex level source $S = \{v_0\}$, fix $d \geq 1$, and let $e \in E(G)$ be an edge lying on the seam $C_T = B^{(T)}_{\text{out}}$ of some tire tread $T \in \mathcal{T}(G, S)$ at depth d. Then there is at most one other tire tread $T' \in \mathcal{T}(G, S)$ at the same depth d with $e \in C_{T'}$.*

Proof. By Theorem 1.18, C_T is the boundary cycle of a bounded face of the parent's inner outerplanar graph $O^{(T_p)}$, where $T_p \in \mathcal{T}(G, S)$ is the parent of T at depth $d-1$. The inner dual of an outerplanar graph is a tree (a forest, if the outerplanar graph is disconnected), so each edge of $O^{(T_p)}$ lies on at most two of its bounded face cycles. Hence e lies on at most one other bounded face cycle of $O^{(T_p)}$, corresponding (Theorem 1.18, child–face bijection) to at most one sibling of T at depth d whose seam contains e. $\square$

Conjecture 1.25 (Seam structure of minimum 4CT counterexamples, sketch)**.** *Suppose the Four Colour Theorem fails: there exists a maximal planar graph that is not* 4*-colourable. Let G be a* minimum *such counterexample (with $|V(G)|$ minimal among non-*4*-colourable maximal planar graphs). Then:*

Restatement-of-classical content.

- (C1) Bilateral colourability. *For every non-trivial seam C of G, both* $\mathrm{Col}(G_T \mid C)$ *and* $\mathrm{Col}(G_C^{\mathrm{ext}} \mid C)$ *are non-empty.*
- (C2) Bilateral incompatibility. *For every non-trivial seam C,*

$$\mathrm{Col}(G_T \mid C) \;\cap\; \mathrm{Col}(G_C^{\mathrm{ext}} \mid C) \;=\; \emptyset.$$

- (C3) Length lower bound (Birkhoff). *Every non-trivial seam C of G has $|V(C)| \geq 6$.*

*(C1) and (C2) together restate "G is a counterexample whose every internal cut by a seam splits into two colourable pieces with incompatible boundary palettes"; (C1) follows from minimality applied to each side after closing the polygonal outer face by a single apex, (C2) from G itself being non-*4*-colourable. (C3) is Birkhoff's internally-*6*-connected condition restated in the seam language.*

Substantive (speculative) content.

- (C4) Innermost obstruction. *There exists a vertex source $v_0 \in V(G)$ and a* leaf *tread $T^* \in \mathcal{T}(G, \{v_0\})$ (a tread with no children in the tree-of-treads) such that:*
  - (i) *the seam interior G_{T^*} is, up to plane iso, one of a finite list of* minimal seam configurations*, characterized by their boundary palette* $\mathrm{Col}(G_{T^*} \mid C_{T^*})$ *being a specific proper subset of the proper* 4*-colourings of the cycle C_{T^*};*
  - (ii) *the path in $\mathcal{T}(G, \{v_0\})$ from the root T_0 to T^* is an* obstruction chain*:* $\mathrm{Col}(G_T \mid C_T)$ *is monotonically restricted (under the natural pull-back along parent–child seams of Remark 1.21) as T descends from the root to T^*, with the final restriction at T^* being incompatible with the v_0-side palette.*

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