Cut-and-depth-label: a procedure for labelling half-graphs of a 6-edge cut by "distance to the cut"

Procedure

Given a maximal planar graph G and its dual G':

1. Find a 6-edge cut $C \subseteq E(G')$ partitioning $V(G')$ into S and $V \setminus S$ with both sides non-empty.
2. Remove the cut edges to obtain two graphs $G'_0 := G'[S] \cup \emptyset_C$ and $G'_1 := G'[V \setminus S] \cup \emptyset_C$ (each a cubic graph minus boundary edges).
3. In each G'_i:
   (a) Let V_i be the set of vertices of degree 2 in G'_i (i.e. original cubic vertices incident to exactly one cut edge; the cut edge accounts for the missing edge, so the vertex sits at degree $3 - 1 = 2$).
   (b) For each $v \in V_i$, attach a new pendant edge to a fresh vertex, and **label these pendant edges with depth** 0.
   (c) For $d = 0, 1, 2, \ldots$: label every unlabelled edge that shares a vertex with a depth-d edge with depth $d + 1$.
   (d) Stop when every edge has a depth label.

Interpretation. The depth labels give a BFS distance in the line graph of G'_i starting from the pendants added at the cut. Equivalently, the depth of an edge e in G'_i is the minimum number of edges traversed (via shared-vertex adjacency) to reach a pendant.
Caveat. If the cut C is not a *matching* cut (i.e., the 6 cut edges share vertices), then some boundary vertices have degree < 2 in G'_i and do not receive a pendant under the strict reading of step 3(a). When the cut is a matching, each of the 12 boundary vertices (6 per side) has degree exactly 2 and receives a pendant; the construction is symmetric.

Example: Holton-McKay graph #0

We apply the procedure to the first of the six non-Hamiltonian 38-vertex cubic plane graphs found by Holton and McKay (loaded from `papers/even_level_graph_generators/experiments/nonham38m4.pc`). This G' is itself a cubic plane graph; its dual G is a 21-vertex triangulation.

The chosen 6-edge cut. Greedy search over 128 distinct 6-edge cuts in G', preferring *matching cuts* (both sides have 6 distinct boundary vertices) and then balanced $|S|$, returns

$$|S| = 10, \qquad C = \{(34, 29),\ (35, 30),\ (26, 22),\ (27, 23),\ (28, 24),\ (31, 25)\}.$$

This is a matching cut: the 6 edges have 12 distinct endpoints, 6 on each side.

Resulting half-graphs.

	G'_0	G'_1
$\lvert S\rvert$	10	28
Original vertices in G'_i	10	28
$\lvert V_i\rvert$ (pendants added)	6	6
Total vertices in G'_i	16	34
Total edges	18	45
Max depth assigned	2	7

Multi-cut vertices (degree < 2 in induced subgraph): *none* on either side, since the cut is a matching.

Visualization.

Cut tires

The depth labelling on G'_i organises its edges into layers indexed by distance to the cut. Each layer gives rise to a family of "cut tires" that play the same structural role as the tires of `paper.tex` but are derived from the cut rather than from a level source in the primal G.

Definition (Cut tire)**.** *Let G'_i be a cut half (Step 3 above) with edge depth labelling* depth : $E(G'_i) \to \mathbb{Z}_{\geq 0}$*. Fix $d > 0$ and let $H_d \subseteq G'_i$ be the subgraph induced on the edges of depth d (vertex set = endpoints of depth-d edges, edges = depth-d edges). Equip H_d with the planar embedding inherited from G'_i.*

For each face f of H_d, the cut tire at (d, f) *is the graph built as follows:*

- *Start with the boundary walk of f as the* face boundary *(every edge here is of depth d in H_d).*
- *For each vertex v on the boundary walk of f that has degree exactly 2 in H_d:*
  - *If v is incident in G'_i to an edge of depth $d-1$, add a fresh vertex n_v and a fresh edge $\{v, n_v\}$, labelled an* out spoke*.*
  - *If v is incident in G'_i to an edge of depth $d+1$, add a fresh vertex n_v and a fresh edge $\{v, n_v\}$, labelled an* in spoke*.*

Convention. *"Out" means toward lower depth (toward the cut, the outer boundary of G'_i); "in" means toward higher depth (deeper into G'_i). Because G'_i is cubic at every original vertex, a degree-2 boundary vertex v has exactly one non-H_d edge in G'_i, so v contributes exactly one spoke (either in or out, not both).*

Structural remark. *Under this definition, each cut tire is intrinsically a graph of the form "cycle (or closed walk) + pendants attached at simple boundary vertices," structurally isomorphic to the partial tire dual $D(T)$ of* `paper.tex` *(Proposition 1.8: $D(T) \cong C_{n+m} \circ K_1$ in the spoke-only case). The labelled in/out spokes are the analogue of $D(T)$'s leaves; the face boundary plays the role of T'_{ann}.*

Relation to the existing tire framework. Under the correspondence between primal level structure (`paper.tex`) and dual depth labelling, a cut tire on G'_i at (d, f) is isomorphic as a graph to the partial tire dual $D(T)$ of a tire T in `paper.tex` (Def. 1.7):

- face boundary at depth $d \longleftrightarrow T'_{\text{ann}}$, the cycle (or closed walk) of annular dual vertices;
- in/out spokes $\longleftrightarrow$ the leaves of $D(T)$ (Def. 1.7's degree-1 pendants attached to the cycle);
- the cut tire's planar embedding $\longleftrightarrow$ $D(T)$'s embedding.

The cut tire is therefore not just analogous to but *is* a partial tire dual (up to relabelling), with depth-from-cut playing the role of level depth from the primal source. Consequently every proposition about $D(T)$ in paper.tex — chromatic polynomial counts, S_3-orbit structure, rainbow conjecture, etc. — applies verbatim to each cut tire.

Example on G'_1 (Holton-McKay #0, $V \setminus S$ half). In this example, counting only degree-2 boundary vertices (which under the redefinition each contribute exactly one in/out spoke):

- $d = 1$: face length 12, 5 out spokes (toward depth-0 pendants) + 0 in spokes. The outermost cut tire, immediately adjacent to the cut. No in spokes because every depth-2 neighbour of a face-boundary vertex is at a higher-degree vertex of H_1.
- $d = 2$: face length 7 (one of two symmetric faces in H_2), 4 out + 3 in spokes.
- $d = 4$: face length 8, 2 out + 1 in.
- $d = 5$: face length 14, 4 out + 2 in.
- $d = 6$: face length 12, 3 out + 2 in. Innermost cut tire in this chain.

A looser chain pigeonhole hypothesis

The rainbow conjecture and König-lift conjecture (from `rainbow_proof.tex` and `worst_case_proof_sketch.tex`) are statements about specific structured face boundaries (antipodal-chord SP). Generic cut tires in our example do not have those specific structures, so those conjectures don't apply directly. Empirically (`cut_tire_conjecture_tests.tex`), cut-tire projections are nonetheless large and well-behaved:

- Every cut tire's joint projection $\pi(T) := \{(\sigma_{\text{out}}, \sigma_{\text{in}}) : \chi \text{ proper}\}$ is S_3-closed (universal, follows from color symmetry of proper edge 3-colouring).
- All S_3-orbits in $\pi(T)$ have size exactly 3 (the constant-colour orbit, when present) or 6 (the generic orbit using all 3 colours); no size-2 stabilizer orbits occur.
- Non-trivial cut tires have $|\pi(T)| \gg 6$: e.g. at depth 2 the projection has size $126 = 21 \cdot 6$ (21 full S_3-orbits), and at depth 6 size $93 = 1 \cdot 3 + 15 \cdot 6$.

This motivates a much weaker conjecture than the rainbow/König ones — one that does not require special face-boundary structure and may be easier to prove:

> **Loose chain pigeonhole conjecture (cut tires, $k \geq 2$ form).** Let $T = T_d^{(i,f)}$ be a cut tire with total spoke count $|\text{in spokes}| + |\text{out spokes}| \geq 2$. Then the joint projection $\pi(T)$ is non-empty, S_3-closed, and contains at least one full S_3-orbit of size 6 (i.e. a σ using ≥ 2 distinct colours), giving $|\pi(T)| \geq 6$.

Consequently, for the chain $T_1^{(i)} \to T_2^{(i)} \to \cdots$ on each side of a cut, the realisable cut-configuration set $\mathcal{R}_i$ contains a full S_3-orbit, and the intersection $\mathcal{R}_0 \cap \mathcal{R}_1$ contains a common S_3-orbit (of size 6), forcing $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$.

Restriction rationale. Cut tires with exactly $k = 1$ in/out spoke have $\pi(T) \subseteq \{1, 2, 3\}$, where every single-colour σ has S_3-stabilizer of order 2, so its S_3-orbit has size $6/2 = 3$, never 6. Such tires trivially violate "$|\pi(T)| \geq 6$" (see `loose_conjecture_counterexamples.tex`, where two $k = 1$ cut tires are exhibited in G_1' of Holton-McKay #0). The $k \geq 2$ restriction excludes these trivially excluded cases.

Justification of the per-tire half. The per-tire half is essentially Proposition 1.13 of `paper.tex`: a spoke-only cut tire (face boundary a simple cycle of length n) has $2^n + 2(-1)^n$ proper edge 3-colourings, which is ≥ 6 whenever $n \geq 3$. By S_3-closure, the orbit count is at least 1, hence ≥ 6 distinct projections (modulo the size-3 constant orbit edge case). For cut tires with non-simple face boundary (e.g. $\theta(1, p, q)$-shape) similar lower bounds follow from the chromatic polynomial of the corresponding $D(T)$ structure (see the menagerie note for explicit closed forms when chord matchings are present).

Justification of the chain half. Chain composition preserves S_3-symmetry (each step is S_3-equivariant), so $\mathcal{R}_i$ inherits S_3-closure from the per-tire projection. The non-trivial claim is that the orbit structure *survives* composition: an S_3-orbit in $\pi(T_d)$ projects to some non-empty S_3-closed subset on the out-spoke side, and this propagates layer-by-layer.

What this would (and would not) prove. If the loose conjecture holds, chain pigeonhole at the cut gives $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$, which (under the minimum- counterexample reduction) contradicts G' being a 4CT counterexample. This is the same final conclusion as the rainbow/König route, with weaker hypotheses but stronger empirical support.

It would *not* prove the rainbow conjecture itself (which is a strict structural claim about the antipodal-chord SP case), nor the König-lift conjecture's specific construction; it would only recover the bottom-line chain pigeonhole result.

Status. Strong empirical support:

- Broader sweep across all 6 Holton-McKay graphs, 3 cuts per graph, both sides (`experiments/loose_conjecture_sweep.py`; data: `loose_conjecture_sweep_data.txt`):
  - 287 total cut tires examined.
  - 154 with $k \geq 2$ in/out spokes.
  - 107 of these had ≤ 14 edges and were brute-force verified; the remaining 47 were too large to enumerate exhaustively.
  - **All** 107 **passed:** $|\pi(T)| \geq 6$ **with a full S_3-orbit.**
  - 0 **counterexamples found.**
- The per-tire half (existence of a multi-color spoke configuration) is provable for spoke-only cut tires (face boundary a simple cycle of length $n \geq 3$) via Prop 1.13:

$2^n + 2(-1)^n$ total proper edge 3-colourings, of which at most 6 are "all-spokes-same-color" (only possible when n is even and the cycle admits a 2-edge-colouring), leaving $\geq 2^n + 2(-1)^n - 6 > 0$ multi-color configurations for $n \geq 3$ even or $n \geq 3$ odd.

- The chain half is the genuinely open question — it requires showing that the per-tire S_3-orbit structure composes coherently through the depth chain.

Connection to chain pigeonhole / 4CT reducibility

The procedure mirrors the 4CT cut-and-reglue scheme (`rainbow_proof.tex`, `worst_case_proof_sketch.tex`, `two_approaches_comparison.tex`) at the structural level. After cutting, each G'_i is a cubic-minus-boundary graph; the pendant additions formally restore cubicity at the 6 degree-2 boundary vertices. The depth label on each edge measures its "distance to the cut."

For a minimum counterexample to the 4CT (i.e., a cubic plane G' with no proper 3-edge-colouring), the depth labels organise each G'_i into concentric layers indexed by distance to the cut. The 3-edge-colourings of G'_i must extend a colouring at the depth-0 pendants (= a ring colouring at the cut); the BFS ordering by depth is the natural induction order for propagating the colouring inward.

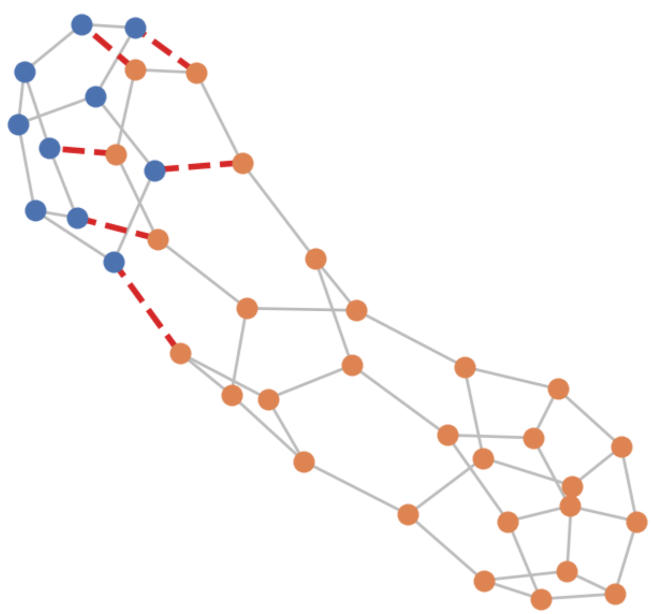
In the tire framework, the cut cycle γ in the primal G (corresponding to the 6-edge cut in G' via planar duality) plays the role of the tire's inner boundary on one side and outer boundary on the other. The depth label on G'_i-edges is exactly the dual analogue of plane depth from γ (cf. the level-cycle generalization discussion in the recent conversation).

Limitations of this example

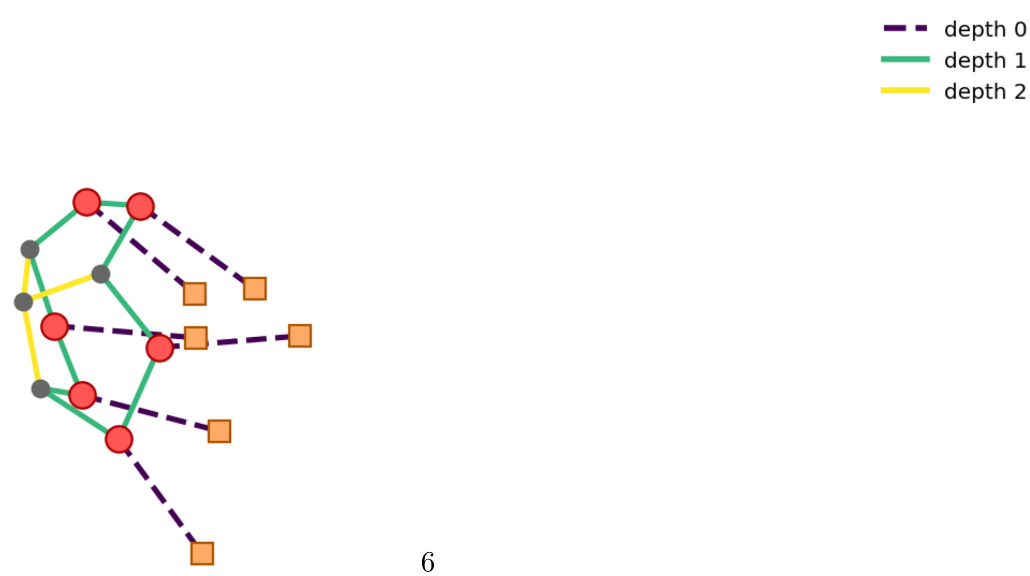
- Holton-McKay graphs are cyclically 5-edge-connected (not 6-edge-connected), so 6-edge cuts are not the minimum cyclic cut. The smallest cyclic edge cuts in this graph are size 5.
- The matching 6-cut found is highly imbalanced ($|S| = 10$ vs. $|S^c| = 28$). Searching among the 128 distinct 6-edge cuts for a balanced matching cut may give better examples.
- Depth labels propagate via the line graph, not via vertex BFS. An alternative procedure would label *vertices* by BFS distance from the boundary; both yield similar layered structures but with slightly different counts.

Cut-and-depth-label procedure on Holton-McKay graph #0
Same vertex positions used across all three panels.

G' = Holton-McKay #0 with 6-edge cut highlighted
Blue = S ($|S| = 10$); orange = $V \setminus S$ ($|V \setminus S| = 28$); red dashed = cut



G'_0 (|S| = 10, |V| = 6, max depth = 2)



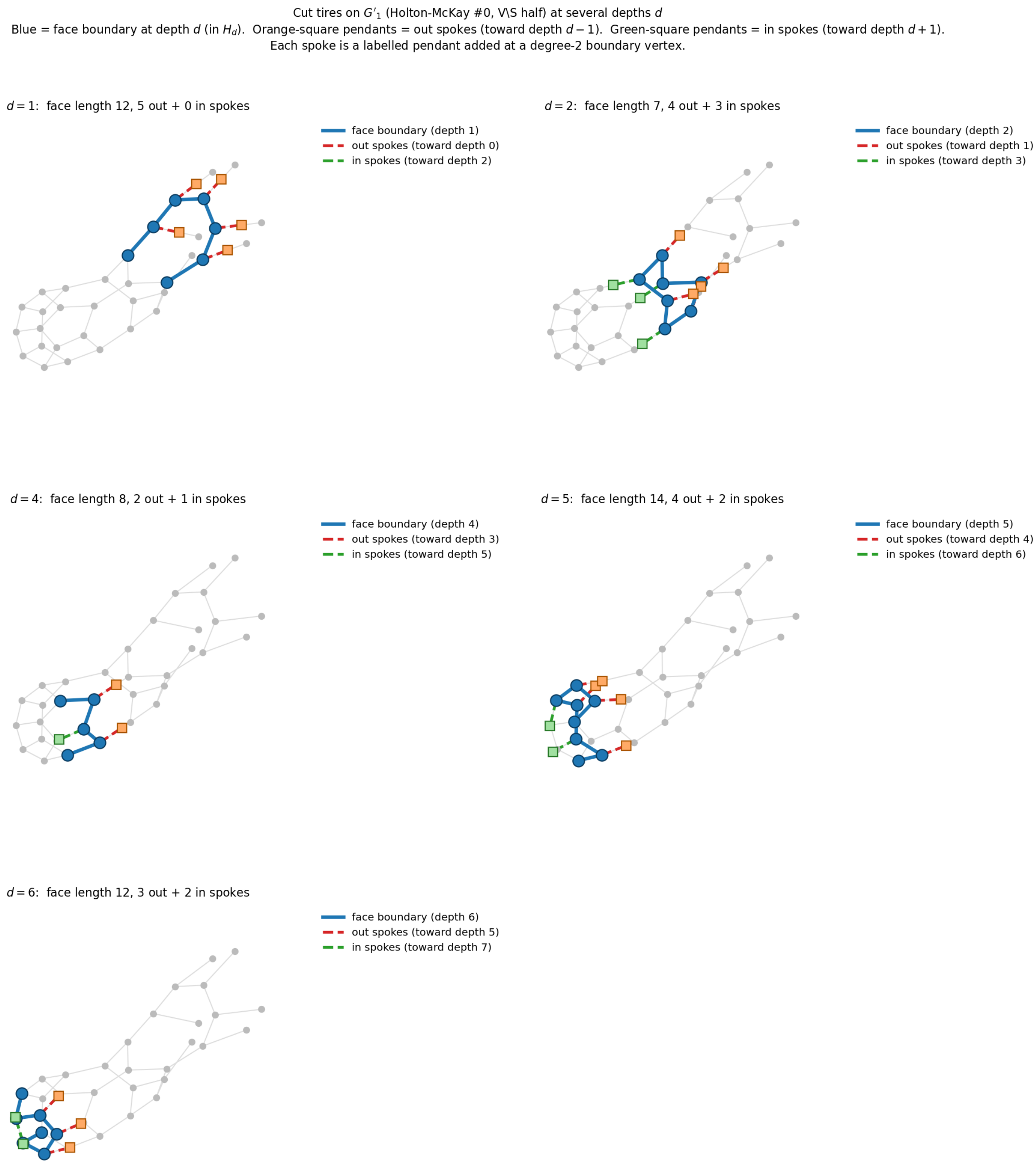


Figure 2: Cut tires on G'_1 at depths $d = 1, 2, 4, 5, 6$. In each panel, blue solid edges form the face boundary at depth d; red dashed edges are inner spokes (depth $d-1$); green dashed edges are outer spokes (depth $d+1$). Vertices on the face boundary are highlighted; the rest of G'_1 is faded.