COLORING NESTED TIRE GRAPHS

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Abstract.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

We study the structure such a minimal counterexample would have to exhibit through the lens of *nested level duals*. Fixing a level source S in G endows the dual G' with a Breadth-First-Search–derived labelling, the dual depth of Definition 1.4, and the level structure of G organises G' into a family of nested cycles carrying these labels. Our aim is to express the obstruction to a 3-edge-colouring of G' as conditions on this nested labelled-cycle structure.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \mathrm{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

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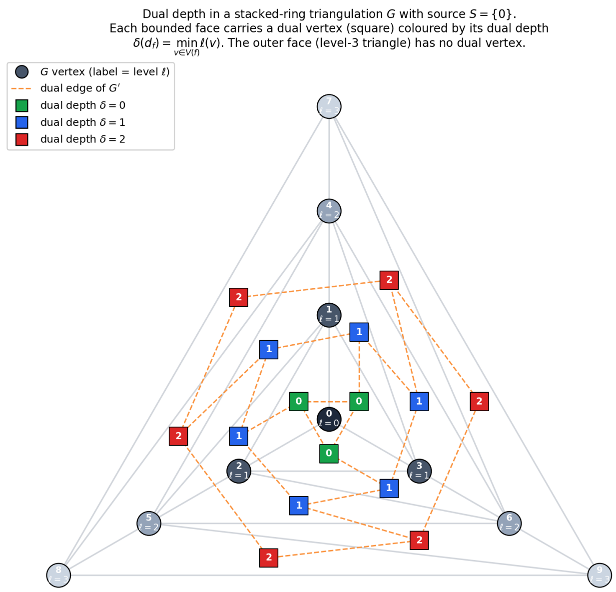

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.5 (Tire graph)**.** A *tire graph* consists of a plane graph T together with two *boundary parts* $B_{\text{out}}, B_{\text{in}} \subseteq T$ and an *inner outerplanar graph* $O \subseteq T$, where each of B_{out} and the outer-face boundary B_{in} of O is either

- a simple cycle of length ≥ 3, or
- a single vertex (a *degenerate* boundary),

with at most one of $B_{\text{out}}, B_{\text{in}}$ degenerate, and $V(B_{\text{out}}) \cap V(O) = \emptyset$. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \qquad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T, the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R. The region R is a closed annulus when both B_{out} and B_{in} are cycles, and a closed disk when exactly one of them is a single vertex.

We call B_{out} the *outer boundary*, O the *inner outerplanar graph*, and B_{in} the *inner boundary* of T. A tire graph in which B_{out} (respectively B_{in}) is a single vertex is said to have a *degenerate outer (respectively inner) boundary*; in either case T is a triangulated closed disk with that vertex as apex.

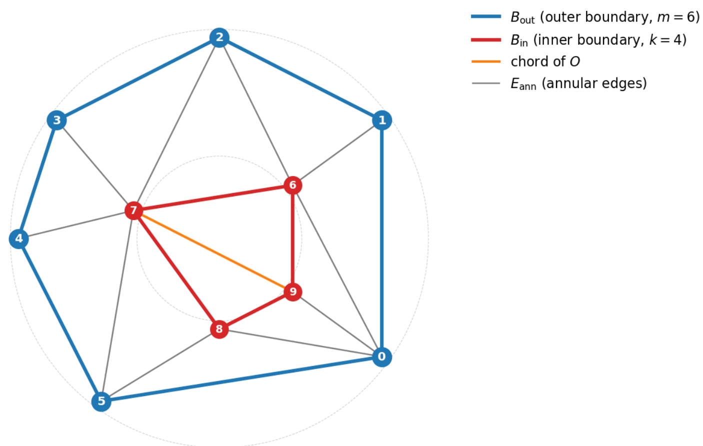

FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \ldots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \ldots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7\text{–}9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Remark 1.6. Let $m = |V(B_{\text{out}})|$ and $k = |V(B_{\text{in}})|$. By Euler's formula on the annular (resp. disk) region R, the tire graph has $m+k$ triangular faces inside R and $|E_{\text{ann}}| = m + k$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(m, k) = 1$), there are $m+k-1$ triangular faces and $|E_{\text{ann}}| = m + k - 1$.

Lemma 1.7 (Tire-component lemma)**.** *Let G be a maximal planar graph and let $S \subseteq V(G)$ be a level source. Fix a plane embedding Π_G of G in which S lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For $d \geq 0$, let*

$$G'_d \;:=\; G'\big[\,\{d_f \in V(G') : \delta_G(d_f) = d\}\,\big]$$

be the inner-dual subgraph on dual vertices of dual depth d, and let C' be a connected component of G'_d. Write $F_{C'} := \{f : d_f \in V(C')\}$, $V_{C'} := \bigcup_{f \in F_{C'}} V(f)$, and let $C := G[V_{C'}]$ inherit its embedding from Π_G. Set $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$.

Assume:

- *(R1) $R_{C'}$ is a topological 2-manifold with boundary; equivalently, at every $v \in V_{C'}$ the faces of $F_{C'}$ incident to v form a single contiguous arc in the rotation around v in Π_G.*
- *(R2) $R_{C'}$ has at most two boundary components.*

Then C, with the inherited embedding, is a tire graph in the sense of Definition 1.5. Its outer boundary B_{out} is the side of $R_{C'}$ closer to S in Π_G, namely the level-d subgraph $G[V_{C'} \cap L_d]$; its inner boundary B_{in} is the side farther from S, namely the level-$(d+1)$ subgraph $G[V_{C'} \cap L_{d+1}]$; and the triangular faces of C inside the closed boundary region are exactly the faces of G in $F_{C'}$.

Proof. Outerplanarity of the two level parts. By construction S lies on the outer face of Π_G, so Lemma 2.6 of [1] applies directly with (G, Π_G, S), giving that $G[L_{d'}]$ is outerplanar for each $d' \geq 0$. Subgraphs of outerplanar graphs are outerplanar, so $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are both outerplanar.

Layer containment. Each $f \in F_{C'}$ has at least one vertex at level d, and adjacent vertices in G differ in level by at most 1; combined with $\delta_G(d_f) = d$, this forces $V(f) \subseteq L_d \cup L_{d+1}$. Hence $V_{C'} \subseteq L_d \cup L_{d+1}$, and C has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$.

Boundary edges are monochromatic in level. Each edge e on $\partial R_{C'}$ separates a face $f \in F_{C'}$ from a face $f' \notin F_{C'}$. Because f and f' share the edge e, their dual vertices are adjacent in G'; if both had depth d they would lie in the same component of G'_d, contradicting $d_f \in C'$ and $d_{f'} \notin C'$. Hence $\delta_G(d_{f'}) \neq d$; combined with the bounded-step property of δ across G'-adjacent faces, $\delta_G(d_{f'}) \in \{d-1, d+1\}$.

- If $\delta_G(d_{f'}) = d - 1$, the third vertex w of $f' = \{u, v, w\}$ (where u, v are the endpoints of e) has $\ell(w) = d - 1$. Each of u, v has $\ell \in \{d, d+1\}$ (from $V(f) \subseteq L_d \cup L_{d+1}$) and is adjacent to w, forcing $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$.
- If $\delta_G(d_{f'}) = d+1$, then all three vertices of f' lie in $L_{\geq d+1}$, so in particular $\ell(u) = \ell(v) = d+1$.

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of $\partial R_{C'}$ is monochromatic in level.

Boundary components are simple cycles. By hypothesis (R1), $R_{C'}$ is a 2-manifold with boundary, so locally at any boundary point p the region $R_{C'}$ is homeomorphic to a half-disk and the link of p in $\partial R_{C'}$ is an arc with two endpoints. In particular, at every boundary vertex v exactly two boundary edges are incident, and the boundary walk traverses v exactly once. Each boundary component is therefore a simple closed walk in G — a simple cycle, possibly degenerating to a single vertex if v has no incident boundary edges (which happens precisely at the BFS endpoints $d = 0$ with $S = \{v_0\}$, or where an entire level set $V_{C'} \cap L_{d+1}$ is empty).

Topological type. $R_{C'}$ is a connected, compact, planar 2-manifold with boundary; planarity gives orientability and genus zero, so by the classification of surfaces $R_{C'}$ is homeomorphic to a closed disk with $n-1$ open disks removed, where $n \geq 1$ is the number of boundary components. Hypothesis (R2) gives $n \leq 2$, so $R_{C'}$ is either a closed disk ($n = 1$) or a closed annulus ($n = 2$).

Tire structure. Because S lies on the outer face of Π_G, the level-d vertices are closer to S in Π_G than the level-$(d+1)$ vertices; in either the annulus or disk case the boundary cycle on the L_d side is the boundary of $R_{C'}$ facing S (the "outer" boundary), and the L_{d+1} side is the boundary facing the interior (the "inner" boundary). This identifies $B_{\text{out}} = G[V_{C'} \cap L_d]$ and $B_{\text{in}} = G[V_{C'} \cap L_{d+1}]$. In the disk case ($n = 1$) one of the two level sets is a single vertex (the BFS endpoint at $d = 0$ with $S = \{v_0\}$, or symmetrically at $d = D_{\max}$ where the inner side collapses to a deepest vertex), giving the degenerate-boundary case of Definition 1.5.

The triangular faces inside the closed boundary region of C are by construction the depth-d faces in $F_{C'}$, and the edges of C are $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$ where E_{ann} are the edges of G between $V_{C'} \cap L_d$ and $V_{C'} \cap L_{d+1}$ that bound a face of $F_{C'}$. $\square$

Remark 1.8. Either boundary part of C in Lemma 1.7 may be degenerate. At $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ as the degenerate *outer* boundary and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G) as the inner boundary. Symmetrically, at $d = D_{\max}$, $V_{C'} \cap L_{D_{\max}+1} = \emptyset$ degenerates to a single deepest vertex serving as the *inner* boundary, with the level-$D_{\max}$ cycle as the outer boundary.

Remark 1.9. The two hypotheses of Lemma 1.7 hold in many natural settings but can fail in general:

(R1) and the pinch obstruction. Hypothesis (R1) fails at a *pinch vertex* $v \in V_{C'}$ when the faces of $F_{C'}$ incident to v split into two or more disjoint arcs of the rotation around v in Π_G. Such a v has at least four neighbours $w_i, w_{i+1}, w_j, w_{j+1}$ (with $i + 1 < j$) in cyclic order such that the faces $\{v, w_i, w_{i+1}\}$ and $\{v, w_j, w_{j+1}\}$ are both depth-d (both endpoints at level $\geq d$) while at least one face in each of the rotation gaps between them carries depth $\neq d$. Concretely, this occurs precisely when the cyclic level sequence $\ell(w_1), \ldots, \ell(w_{\deg v})$ enters and leaves $\{d, d+1\}$ more than once. Whenever such a v exists and the two arcs are joined to a common component of G'_d by some *other* path of depth-d faces (not through v), the resulting $R_{C'}$ is a wedge of two manifold regions at v, violating (R1).

(R2) and the multi-hole obstruction. Hypothesis (R2) fails when the depth-d region $R_{C'}$ encloses two or more disjoint depth-$> d$ sub-regions. In a BFS the depth-$< d$ region (the BFS ball of radius $d - 1$) is connected, so at most one boundary component of $R_{C'}$ can lie on the source side; (R2) is therefore equivalent to "the closure of the depth-$> d$ region adjacent to $R_{C'}$ has at most one connected component." Multi-hole topology arises when several disjoint depth-$> d$ "lobes" of G sit inside the same depth-d component.

In the special case $d = 0$ with single-vertex source $S = \{v_0\}$ both hypotheses hold automatically: $R_{C'}$ is the star of v_0, a topological closed disk with one boundary cycle (the link of v_0), giving a tire graph with degenerate inner boundary $\{v_0\}$.

References

[1] E. Bauerfeld, *Plane Depth Sequencing*, manuscript (math-research repository), 2026.