

# The boundary cut tire $T_\partial$ : closing the coverage gap

## Motivation

The high-side cut tire forest (`cut_tire_tree_structure.tex`) omits the low-side face of  $H_1$  — the unique face containing the pendants of  $G'_i$ . This omission is essential to prove the forest’s tree structure (low-side faces span multiple parent faces of  $H_{d-1}$ , violating the uniqueness step in the proof).

**The coverage gap.** Empirically (`chain_dp_joint.py` on the dodecahedron, cut #0, side 0): when  $|S_i|$  is small,  $H_1$  on side  $i$  can be a *tree* (no cycles). Its unique face is forced to be low-side (contains pendants), so the high-side forest is *empty*. The chain DP, projecting through roots that don’t exist, gives  $\mathcal{R}_i = \emptyset$  — but  $G$  may still be 3-edge-colourable, with non-empty  $\mathcal{R}_i$ .

**Resolution.** Introduce the *boundary cut tire*  $T_\partial^{(i)}$  on each side  $i$ , representing the low-side face of  $H_1$  together with the depth-0 pendants. It is not a child of any other tire (it sits at the framework’s outer boundary), but the chain DP can use it as a special root that interfaces between the high-side forest interior and the cut.

## Definition

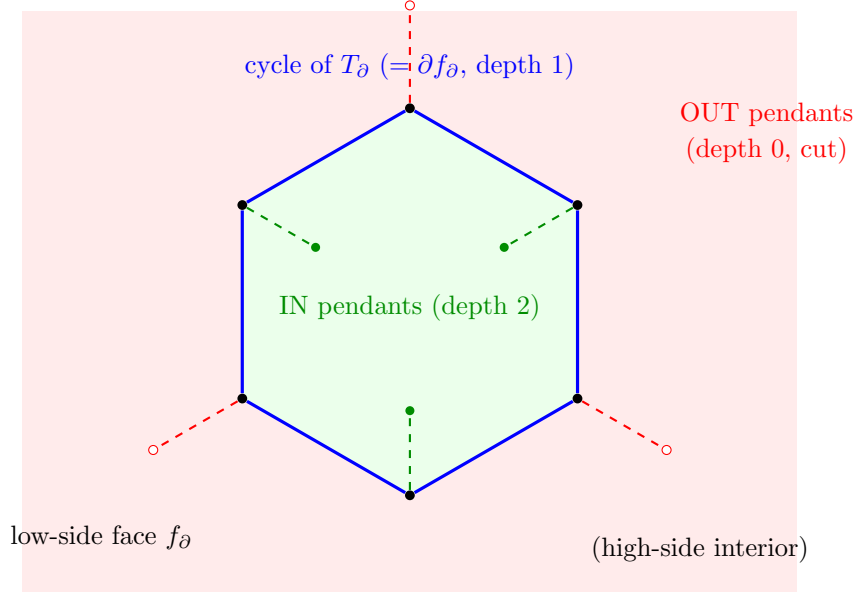
**Setup.** Fix a side  $i$  of a 6-edge cut of  $G$ . Let  $G'_i = (G[S_i] \cup \text{pendants})$  with edge depths assigned by BFS from pendants (depth 0).  $H_d$  = the subgraph of  $G'_i$  spanned by depth- $d$  edges.

**The low-side face of  $H_1$ .** By the level-set lemma, each face of  $H_1$  is entirely low-side (interior contains only depth-0 edges = pendants) or entirely high-side (interior contains only depth- $\geq 2$  edges). There is exactly one low-side face: the unique face whose interior contains the pendants. Call it  $f_\partial^{(i)}$ .

**Definition** (Boundary cut tire  $T_\partial^{(i)}$ ). *The boundary cut tire on side  $i$  is the labelled multigraph  $T_\partial^{(i)}$  obtained from:*

- *The boundary walk of  $f_\partial^{(i)}$  in  $H_1$  (the edges of  $H_1$  on the boundary of the low-side face), with each depth-1 edge contributing one cycle edge.*
- *For each boundary vertex  $v$  of  $f_\partial^{(i)}$  with  $\deg_{H_1}(v) = 2$ :*
  - *The pendant edge at  $v$  (= depth-0 cut edge), labelled “OUT.”*
  - *If the third edge at  $v$  has depth 2 (= edge of  $H_2$  in an adjacent high-side  $H_1$  face), one labelled “IN” pendant.*

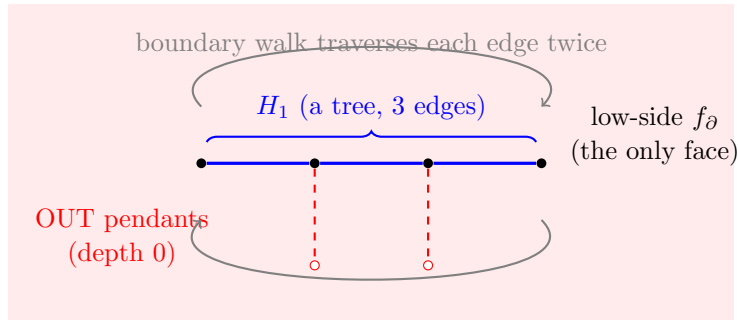
**Picture.** A thick-side example with  $H_1$  a hexagonal cycle in  $G'_i$ : the low-side face  $f_\partial$  is the outer (unbounded) region containing 3 pendants (depth 0); the adjacent high-side face is inside the hexagon, containing depth-2 edges.



The cycle of  $T_\partial$  is the hexagon (blue, depth 1); pendants fall in two classes: OUT (red, dashed, going into the low-side face) attach to the cut, and IN (green, dashed, going into the adjacent high-side region) attach to depth-2 edges shared with  $T_2^{(f')}$  tires.

**What  $T_\partial$  looks like in special cases.**

- *Side with thick BFS:*  $H_1$  has multiple faces, one low-side ( $f_\partial$ ) and one or more high-side.  $T_\partial$  has cycle = boundary of  $f_\partial$  + OUT pendants (= cut edges incident to  $f_\partial$ 's boundary) + IN pendants (= depth-2 edges in adjacent high-side faces).
- *Side with thin BFS* (e.g.  $H_1$  a tree): The unique  $H_1$  face is  $f_\partial$ .  $T_\partial$  has cycle = boundary walk of this single face (each  $H_1$  edge contributes *two* cycle-edge traversals, since the walk goes around each edge on both sides) + OUT pendants at  $V_{\deg=2}$ . No IN pendants (no depth-2 edges exist).

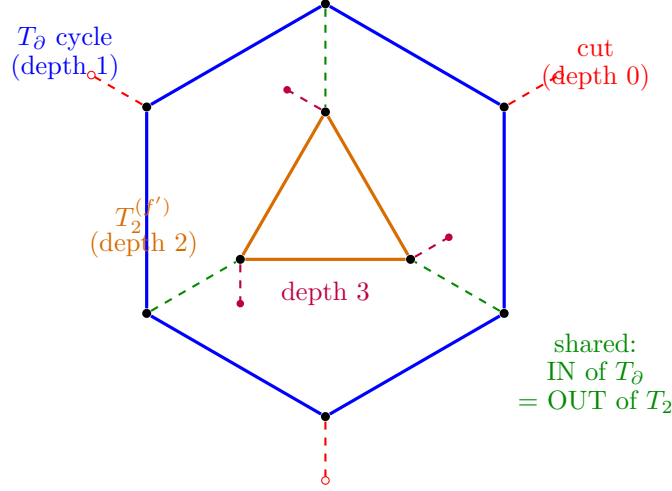


In the thin case (e.g. dodecahedron cut #0 side 0 with  $|S_0| = 4$ ), the boundary walk of the single face has length 6 (each of 3  $H_1$  edges visited twice), and OUT pendants attach at the two  $V_{\deg=2}$  vertices.

## High-side cut tires alongside $T_\partial$

A high-side cut tire  $T_d^{(f)}$  at depth  $d$  has the same structural shape as  $T_\partial$ :

- Cycle = depth- $d$  edges in  $\partial f$  ( $H_d$  edges).
- OUT spokes = depth- $(d - 1)$  edges at boundary vertices (pointing outward, toward parent).
- IN spokes = depth- $(d + 1)$  edges at boundary vertices (pointing inward, toward children).



The green dashed edges are the *shared* depth-2 edges: they are IN pendants of  $T_\partial$  and OUT spokes of  $T_2^{(f')}$ . In any global 3-edge coloring of  $G'_i$ , the color assigned to each green edge is the same when viewed from either tire. The chain DP exploits exactly this shared-edge identity.

## The extended forest

The high-side cut tire forest of  $G'_i$  (proven in `cut_tire_tree_structure.tex`) has roots  $T_1^{(f_{\text{high}})}$  for the high-side faces of  $H_1$ .

**Definition** (Extended cut tire structure). *The extended cut tire structure on side  $i$  is the high-side forest with  $T_\partial^{(i)}$  adjoined as a boundary node.  $T_\partial$  is not a child or parent of any high-side cut tire in the geometric containment sense, but it shares edges with adjacent high-side tires:*

- $T_\partial$  shares depth-1 edges with each high-side depth-1 tire  $T_1^{(f_{\text{high}})}$  whose face  $f_{\text{high}}$  is adjacent to  $f_\partial$  (= shares a boundary edge of  $H_1$ ).
- $T_\partial$ 's IN pendants are cycle edges of depth-2 cut tires  $T_2^{(f')}$  in adjacent high-side  $H_1$  faces.

**Role in the chain DP.** In the joint-projection chain DP, the per-tire state at  $T_\partial$  is a set of valid edge 3-colorings of  $T_\partial$ 's structure (cycle + OUT + IN pendants). Composition with adjacent tires is via shared edges (same  $G'_i$  edge tuple appearing in both), exactly as for the high-side forest. The chain DP then has  $T_\partial$  as its *boundary interface*: the OUT pendants project to the cut, and the IN pendants + cycle edges propagate constraints to/from the high-side forest.

## Chain DP with $T_\partial$ , sketched

1. Process the high-side forest bottom-up as before, producing per-tire valid coloring sets  $A(T)$ .
2. Compute the valid coloring set  $A(T_\partial^{(i)})$  via enumeration of proper 3-edge-colorings on  $T_\partial$ 's structure.
3. Restrict  $A(T_\partial)$  by edge-sharing with adjacent high-side tires:
  - For each  $T_1^{(f_{\text{high}})}$  sharing depth-1 edges with  $T_\partial$ : keep  $T_\partial$  colorings consistent with some  $A(T_1^{(f_{\text{high}})})$  coloring on shared edges.
  - For each  $T_2^{(f')}$  sharing a depth-2 edge with  $T_\partial$ 's IN pendants: keep  $T_\partial$  colorings consistent with some  $A(T_2^{(f')})$  coloring on the shared edge.
4. Project  $A(T_\partial^{(i)})$  to OUT pendants  $= \mathcal{R}_i$ .

## Where things stand

**What this closes.** The coverage gap (no high-side cut tires on thin sides) is resolved:  $T_\partial$  always exists and provides the interface to the cut.

**What it leaves open.**  $T_\partial$ 's “cycle” is the boundary walk of  $f_\partial$ , which is a closed walk in  $H_1$  — *not* necessarily a simple cycle when  $H_1$  is a tree (the walk traverses each edge twice). The per-tire half (Prop 1.13) does not directly apply to such walks. This is a special case of the “branched cut tires” open problem in `chain_half_analysis.tex`.

**Why the chain DP can still work.** Even without the per-tire half guaranteeing  $|\pi(T_\partial)| \geq 6$ , we can *compute* the valid coloring set for  $T_\partial$  directly (brute-force or constraint-propagated enumeration). If this set is non-empty after edge-sharing restrictions, the chain DP yields a non-empty  $\mathcal{R}_i$ .

### Logical status of the extended framework.

- Edge-sharing chain DP  $S_3$ -equivariance: still holds (uniform  $S_3$  action commutes with edge constraints).
- Tree structure: the high-side forest still forms a forest;  $T_\partial$  adjoined is no longer a tree but a connected structure (forest + boundary node).
- Per-tire half for  $T_\partial$ : open, special case of branched tires.
- Non-emptiness of  $\mathcal{R}_i$  via the DP: open; this is the *primary* chain-half claim, now testable with  $T_\partial$  included.

## Empirical next step

Re-run `chain_dp_joint.py` with  $T_\partial$  added. Compare with ground truth (brute-force  $G'_i$  3-edge colorings).

For the dodecahedron cut #0 side 0:

- Ground truth:  $|\mathcal{R}_{\text{ground}}| = 36$ .
- Old high-side-only DP:  $|\mathcal{R}_0| = 0$  (no high-side tires).
- New  $T_\partial$ -extended DP: should match ground truth (or reveal another gap to investigate).