COLORED EDGE FLIP CLASSES

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Abstract.

1. Motivation

The Four Color Theorem asserts that every planar graph is properly 4-colorable, or equivalently that no maximal planar graph G satisfies $\chi(G) \geq 5$. Suppose, towards a contradiction, that such a graph exists; let G_0 be one of minimum order. Any structural property shared by every maximal planar graph H with $|V(H)| = |V(G_0)|$ is then automatically inherited by G_0, and any property *not* satisfied by G_0 excludes a portion of the class of maximal planar graphs from playing the role of a minimum counterexample.

Our principal observation (Theorem 4.4) is that every graph in the *flip neighborhood* of G_0 — the set $\mathcal{N}(G_0)$ of maximal planar graphs obtainable from G_0 by a single admissible edge flip — is 4-colorable. In other words, G_0 sits at the boundary of 4-colorability: a single flip in any direction yields a 4-colorable graph. As an immediate corollary, no such G_0 is *flip-symmetric*, where we call a maximal planar graph G flip-symmetric when some admissible flip at an edge of G returns a graph isomorphic to G; if any flip of G_0 were to return G_0, that flip would witness G_0 as 4-colorable. The search for a counterexample to the Four Color Theorem may therefore be confined to the complement of the class $\mathcal{F}$ of flip-symmetric maximal planar graphs.

To track this rigidity at the level of individual 4-colorings, we introduce the *colored edge flip class* $\mathcal{C}(H, \varphi)$ of a maximal planar graph H and a proper 4-coloring φ of H: the set of maximal planar graphs reachable from H by sequences of admissible edge flips that each preserve φ. Theorem 4.5 records that $G_0 \notin \mathcal{C}(H, \varphi)$ for any $H \in \mathcal{N}(G_0)$ and any proper 4-coloring φ of H.

2. Preliminaries

Let G be a maximal planar graph with $|V(G)| \geq 4$, embedded in the plane so that every face — including the outer face — is a triangle. Every edge $uv \in E(G)$ is then shared by exactly two triangular faces uvw and uvx whose union is a quadrilateral $uwvx$ with diagonal uv.

Definition 2.1 (Edge flip)**.** Let G be a maximal planar graph and let $uv \in E(G)$ be an edge whose two incident triangular faces are uvw and uvx. The *edge flip* (or *diagonal flip*) at uv is the operation that deletes the edge uv and inserts the edge wx in its place, replacing the two triangles uvw and uvx by the two triangles uwx and vwx. The flip is *admissible* if $wx \notin E(G)$; otherwise the resulting multigraph is not simple and the flip is forbidden.

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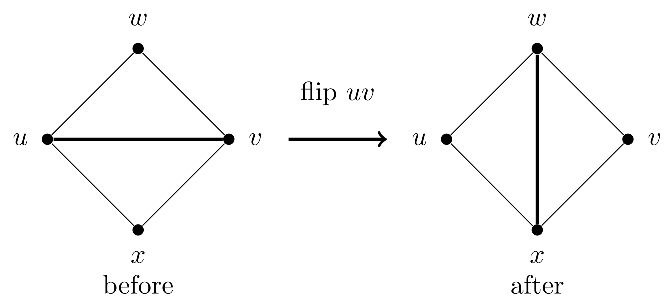

FIGURE 1. An edge flip replaces the diagonal uv of the quadrilateral $uwvx$ with the diagonal wx.

3. Flip-symmetric maximal planar graphs

For a maximal planar graph G and an admissible edge $uv \in E(G)$ with incident triangles uvw, uvx, write

$$G^{\mathrm{flip}(uv)} \;=\; \big(V(G),\, (E(G) \setminus \{uv\}) \cup \{wx\}\big)$$

for the graph obtained from G by flipping uv.

Definition 3.1 (Flip-symmetric graph)**.** A maximal planar graph G is *flip-symmetric* if there exists an admissible edge $uv \in E(G)$ such that $G^{\mathrm{flip}(uv)} \cong G$. We write $\mathcal{F}$ for the class of flip-symmetric maximal planar graphs.

Definition 3.2 (Flip neighborhood)**.** Let G be a maximal planar graph. The *flip neighborhood* of G is the set

$$\mathcal{N}(G) \;=\; \big\{\, G^{\mathrm{flip}(uv)} : uv \in E(G) \text{ and the flip at } uv \text{ is admissible} \,\big\}$$

of maximal planar graphs obtainable from G by a single admissible edge flip.

Definition 3.3 (Colored edge flip class)**.** Let G be a maximal planar graph and let φ be a proper 4-coloring of G. The *colored edge flip class* of (G, φ) is the set $\mathcal{C}(G, \varphi)$ of maximal planar graphs reachable from G by some (possibly empty) sequence of admissible edge flips, each of which leaves φ a proper 4-coloring of the resulting graph. Explicitly, $H \in \mathcal{C}(G, \varphi)$ iff there exist graphs $G = G_0, G_1, \ldots, G_k = H$ such that for each $0 \le i < k$, $G_{i+1} = G_i^{\mathrm{flip}(u_i v_i)}$ for some $u_i v_i \in E(G_i)$ whose flip is admissible in G_i and whose opposite vertices w_i, x_i satisfy $\varphi(w_i) \neq \varphi(x_i)$.

4. The flip neighborhood of a minimum-order counterexample

Definition 4.1 (Edge-deletion subgraph)**.** Let G be a maximal planar graph and $uv \in E(G)$. The *edge-deletion subgraph at* uv is the spanning subgraph $G - uv = (V(G),\, E(G) \setminus \{uv\})$. Write $\mathcal{D}(G) = \{G - uv : uv \in E(G)\}$.

Lemma 4.2. *Let G_0 be a maximal planar graph of minimum order with $\chi(G_0) \ge 5$. Then every $H \in \mathcal{D}(G_0)$ is 4-colorable.*

Proof. Fix $uv \in E(G_0)$ and let G_0/uv denote the simple planar graph obtained by contracting uv and discarding parallel edges. Then G_0/uv is a simple planar graph on $|V(G_0)| - 1 \ge 4$ vertices but is not in general a triangulation; triangulate its planar embedding (by adding chords inside any non-triangular face) to obtain a maximal planar graph T on the same vertex set, with G_0/uv as a spanning subgraph and $|V(T)| < |V(G_0)|$. By the minimality of G_0, T admits a proper 4-coloring, which restricts to a proper 4-coloring c of G_0/uv. Let z be the contracted vertex and define $c' \colon V(G_0) \to \{1,2,3,4\}$ by $c'(u) = c'(v) = c(z)$ and $c'(y) = c(y)$ for $y \notin \{u, v\}$. Every edge of $G_0 - uv$ is either disjoint from $\{u, v\}$ or incident to exactly one of them; in either case the corresponding edge of G_0/uv has distinct endpoints under c, so c' assigns its endpoints distinct colors. The edge uv itself is absent from $G_0 - uv$, so c' is a proper 4-coloring of $G_0 - uv$. □

Lemma 4.3. *Let G_0 be a maximal planar graph of minimum order with $\chi(G_0) \geq 5$, fix $uv \in E(G_0)$, and let φ be any proper 4-coloring of $G_0 - uv$. Write $a = \varphi(u)$ and let b, c, d denote the three remaining colors. Then:*

1. *$\varphi(v) = a$;*
2. *the subgraph of $G_0 - uv$ induced by the vertices of color a or b contains a path from u to v;*
3. *the subgraph of $G_0 - uv$ induced by the vertices of color a or c contains a path from u to v.*

Proof. (1) If $\varphi(v) \neq a$ then φ is already a proper 4-coloring of G_0, since the only edge of G_0 absent from $G_0 - uv$ is uv and its endpoints have distinct colors. This contradicts $\chi(G_0) \geq 5$, so $\varphi(v) = a$.

(2) Suppose, for contradiction, that u and v lie in distinct connected components of the subgraph of $G_0 - uv$ induced by the color classes a and b. Let C be the component containing u, and define $\varphi' \colon V(G_0) \to \{a, b, c, d\}$ by swapping colors $a \leftrightarrow b$ on C and leaving every other vertex unchanged. Then φ' is a proper 4-coloring of $G_0 - uv$ with $\varphi'(u) = b$ and $\varphi'(v) = a$, contradicting part (1) applied to φ'.

(3) Identical to (2) with c in place of b. □

Theorem 4.4. *Let G be a minimum-order maximal planar graph with $\chi(G) \geq 5$. Then every $H \in \mathcal{N}(G)$ is 4-colorable.*

Proof. Fix an edge $e = uv \in E(G)$, and let F_0, F_1 be the two triangular faces of G incident to e, so that $\{w, x\} = \big(V(F_0) \cup V(F_1)\big) \setminus \{u, v\}$. By Lemma 4.2, $G - e$ admits a proper 4-coloring φ.

Case 1: $\varphi(w) \neq \varphi(x)$. Then φ is also a proper 4-coloring of the graph induced by the edge flip of e.

Case 2: $\varphi(w) = \varphi(x)$. Set $a = \varphi(u)$; by Lemma 4.3(1), $\varphi(v) = a$ as well, and the edges $uw, vw \in E(G - e)$ force $\varphi(w) \neq a$. Choose a color $b \notin \{a, \varphi(w)\}$. By Lemma 4.3, there is a path P from u to v in the subgraph of $G - e$ induced by the vertices of color a or b. Let $\{c, d\} = \{1,2,3,4\} \setminus \{a, b\}$; then $\varphi(w) = \varphi(x) \in \{c, d\}$.

Any path from w to x in the subgraph of $G - e$ induced by the vertices of color c or d would, in the plane embedding of $G - e$, cross P; but its vertices have colors in $\{c, d\}$ and the vertices of P have colors in $\{a, b\}$, and these sets are disjoint, so the two paths share no vertex. Hence w and x lie in distinct connected components of the $\{c, d\}$-colored subgraph of $G - e$. Swapping colors $c \leftrightarrow d$ on the component containing w yields a proper 4-coloring of $G - e$ in which $\varphi(w) \neq \varphi(x)$, reducing to Case 1. □

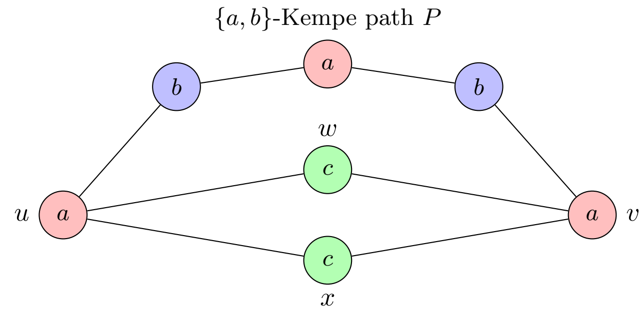

Figure 2. Case 2 of the proof of Theorem 4.4: u, v share color a and w, x share color c. The $\{a, b\}$-Kempe path P from u to v separates w from x in the plane, so no $\{c, d\}$-path between w and x can avoid crossing P; since the color sets $\{a, b\}$ and $\{c, d\}$ are disjoint, no such path exists.

Theorem 4.5. *Let G be a minimum-order maximal planar graph with $\chi(G) \geq 5$. Then for every $H \in \mathcal{N}(G)$ and every proper 4-coloring φ of H,*

$$G \notin \mathcal{C}(H, \varphi).$$

Proof. Suppose, for contradiction, that $G \in \mathcal{C}(H, \varphi)$ for some $H \in \mathcal{N}(G)$ and some proper 4-coloring φ of H. By Definition 3.3, there exists a sequence of maximal planar graphs $H = H_0, H_1, \ldots, H_k = G$ in which each H_{i+1} is obtained from H_i by an admissible edge flip that leaves φ a proper 4-coloring of H_{i+1}. By induction on i, φ is a proper 4-coloring of every H_i; in particular, φ is a proper 4-coloring of $H_k = G$. But $\chi(G) \geq 5$ admits no such coloring, a contradiction. $\square$