

NESTED LEVEL DUALS

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ABSTRACT.

1. INTRODUCTION

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

We study the structure such a minimal counterexample would have to exhibit through the lens of *nested level duals*. Fixing a level source S in G endows the dual G' with a Breadth-First-Search-derived labelling, the dual depth of Definition 1.4, and the level structure of G organises G' into a family of nested cycles carrying these labels. Our aim is to express the obstruction to a 3-edge-colouring of G' as conditions on this nested labelled-cycle structure.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G . We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source). A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 1.2 (Levels). Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \text{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 1.3 (Dual). The *dual* of G , written G' , is the inner (weak) planar dual of G with respect to the embedding Π_G : it has one vertex d_f for each bounded face f of G , and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f' . The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G , and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth). Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \text{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f .

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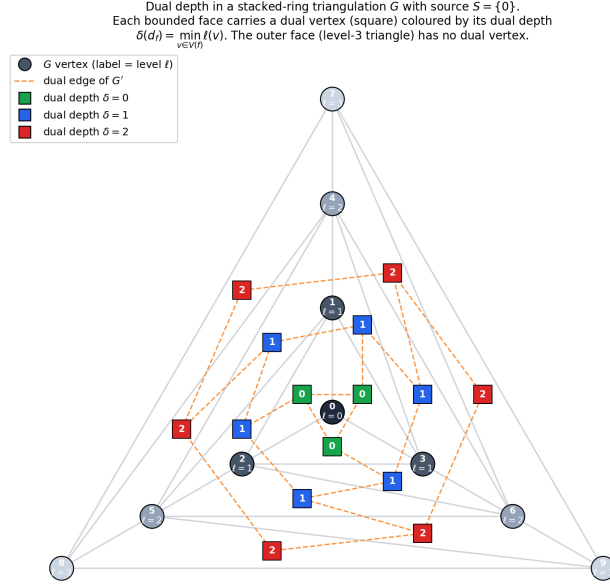


FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ . Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.