

# COLORING NESTED TIRE GRAPHS

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ABSTRACT. We establish the foundational structure of nested level-induced tire decompositions of a plane triangulation  $G$ . A *level source* of  $G$  induces a BFS layering of  $G$  and endows the inner planar dual  $G'$  with a *dual depth* grading. The basic object of study is the *tire graph*  $T$  — a plane graph whose outer and inner boundaries bound a closed planar region, the *tire tread*  $R$ , triangulated by the *annular edges*  $E_{\text{ann}}$ . Our main structural result, the *tire-component lemma*, exhibits each connected component of  $G'_d$  as a tire graph; the *tire-tread partition theorem* consequence shows the resulting tire treads partition the bounded faces of  $G$ . Coloring questions on  $G$  thereby factor through coloring questions on the individual treads.

## 1. INTRODUCTION

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation  $G$  is properly 4-vertex-colourable if and only if its dual cubic graph  $G'$  is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem — a smallest triangulation admitting no proper 4-colouring — corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

The structural study of such a minimal counterexample is the overarching motivation for the present line of work. This first paper establishes the foundational vocabulary — level sources, dual depth, tire graphs, and partial tire duals — on which subsequent papers in the series build. In particular, the companion paper [2] uses these definitions to develop nested-cycle structure theorems and chain-pigeonhole conjectures for tire annular subgraphs of  $G'$ .

Throughout,  $G = (V, E)$  is a plane maximal planar graph (a triangulation) with a fixed planar embedding  $\Pi_G$ . We write  $|V| = n$ , so  $|E| = 3n - 6$  and  $G$  has  $2n - 4$  triangular faces.

**Definition 1.1** (Level source). A *level source* of  $G$  is any vertex  $v \in V$ ; we write  $S = \{v\}$  for the level-0 source.

**Definition 1.2** (Levels). Given a level source  $S \subseteq V$ , the *level* of  $v \in V$  is  $\ell_G(v) = \text{dist}_G(v, S)$ , the graph distance from  $v$  to the nearest source vertex.

**Definition 1.3** (Dual). The *dual* of  $G$ , written  $G'$ , is the inner (weak) planar dual of  $G$  with respect to the embedding  $\Pi_G$ : it has one vertex  $d_f$  for each bounded face  $f$  of  $G$ , and an edge joining  $d_f$  and  $d_{f'}$  for each edge of  $G$  shared by two bounded faces  $f$  and  $f'$ . The unbounded outer face contributes no vertex, and edges of  $G$  on the outer boundary contribute no dual edge. Since  $G$  is a triangulation, each

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vertex  $d_f \in V(G')$  corresponds to a triangular face  $f$  of  $G$ , and we write  $V(f) \subseteq V$  for its three incident vertices.

**Definition 1.4** (Dual depth). Given a level source  $S \subseteq V$ , the *dual depth* of a dual vertex  $d_f \in V(G')$  is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \text{dist}_G(v, S),$$

the smallest level among the three vertices of  $G$  bounding the face  $f$ .

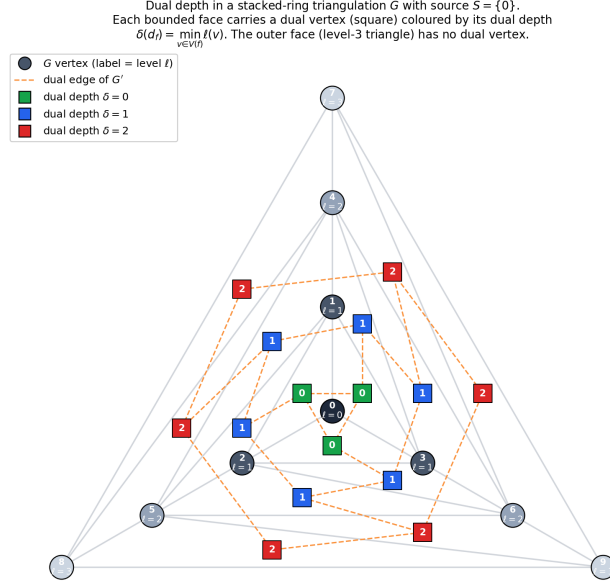


FIGURE 1. Dual depth in a stacked-ring triangulation  $G$  with level source  $S = \{0\}$ . Each  $G$  vertex is labelled by its level  $\ell$ . Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth  $\delta(d_f) = \min_{v \in V(f)} \ell(v)$ : the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

**Definition 1.5** (Tire graph). A *tire graph* consists of a plane graph  $T$  together with an *outer boundary*  $B_{\text{out}} \subseteq T$  and an *inner outerplanar graph*  $O \subseteq T$  with  $V(B_{\text{out}}) \cap V(O) = \emptyset$ , where

- $B_{\text{out}}$  is either a simple cycle of length  $\geq 3$  or a single vertex (a *degenerate outer boundary*);
- $O$  is an outerplanar graph; its *inner boundary*  $B_{\text{in}}$  is the closed walk in  $O$  that traces the boundary of  $O$ 's outer face in the inherited embedding, which is a simple cycle when  $O$  is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if  $|V(O)| = 1$ , we say  $T$  has a *degenerate inner boundary*.

At most one of  $B_{\text{out}}, B_{\text{in}}$  may be degenerate. The vertex and edge sets of  $T$  are

$$V(T) = V(B_{\text{out}}) \cup V(O), \quad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where  $E_{\text{ann}}$  — the *annular edges* — has the property that, in the plane embedding of  $T$ , the closed planar region  $R$  bounded externally by  $B_{\text{out}}$  and internally by  $B_{\text{in}}$  is partitioned into triangular faces of  $T$  whose union is  $R$ . We call  $R$  the *tire tread* of  $T$ .

When  $B_{\text{out}}$  is a simple cycle and  $O$  is 2-connected, the tread is a closed annulus. More generally,  $R$  is a closed planar region that may fail to be a 2-manifold at cut-vertices of  $O$  (where two “lobes” of the depth- $d$  region meet at a single vertex); the inner boundary  $B_{\text{in}}$  is then a non-simple closed walk that visits the cut-vertex multiple times. The relaxed definition accommodates outerplanar inner graphs with bridges, cut-vertices, or multiple connected components. When either boundary is degenerate, the tread is a closed disk with that vertex as apex.

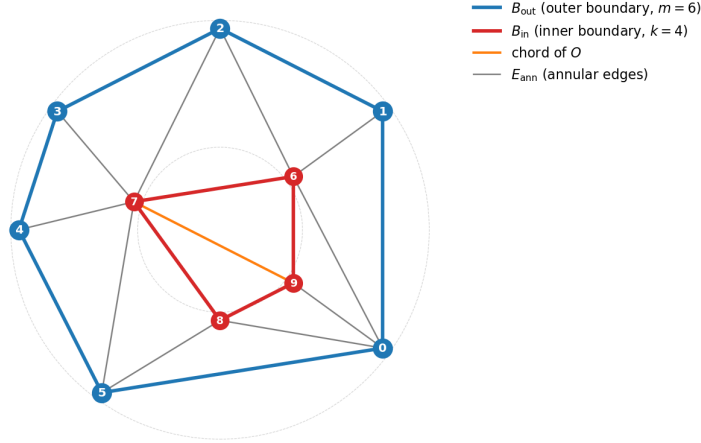


FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary  $B_{\text{out}}$  a 6-cycle on vertices  $0, \dots, 5$  (blue), inner boundary  $B_{\text{in}}$  a 4-cycle on vertices  $6, \dots, 9$  (red), inner outerplanar graph  $O = B_{\text{in}} \cup \{7-9\}$  (with one chord, orange), and  $E_{\text{ann}}$  (grey) tiling the annulus between  $B_{\text{out}}$  and  $B_{\text{in}}$  by ten triangular faces.

*Remark 1.6.* Let  $m = |V(B_{\text{out}})|$  and  $k = |V(B_{\text{in}})|$ . By Euler’s formula on the tire tread  $R$ , the tire graph has  $m + k$  triangular faces inside  $R$  and  $|E_{\text{ann}}| = m + k$  annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so  $\min(m, k) = 1$ ), there are  $m + k - 1$  triangular faces and  $|E_{\text{ann}}| = m + k - 1$ .

**Proposition 1.7** (Source-side simple-cycle property). *Let  $G$  be a maximal planar graph with planar embedding  $\Pi_G$  and single-vertex source  $v_0$ . Let  $d \geq 1$ ,  $v \in L_d$ , and let  $C'$  be a connected component of  $G'_d$  such that  $v$  is incident to some face in  $F_{C'}$ . Then the depth- $d$  faces in  $F_{C'}$  incident to  $v$  form a single contiguous arc in  $v$ ’s rotation in  $\Pi_G$ .*

*Equivalently: for any such component, the source-side boundary of  $R_{C'}$  is a simple cycle in  $L_d$  (no cut-vertices at level  $d$ ).*

*Proof.* Suppose for contradiction that the depth- $d$  faces in  $F_{C'}$  at  $v$  lie in two or more disjoint arcs of  $v$ 's rotation. Adjacent vertices in  $G$  differ in level by at most 1, so a face at  $v$  has depth exactly  $d$  iff both other vertices have level  $\geq d$ , and depth  $\leq d-1$  iff at least one has level  $d-1$ . Hence the gaps between the depth- $d$  arcs at  $v$  are populated by level- $(d-1)$  neighbours of  $v$ , occurring in at least two disjoint arcs of  $v$ 's rotation. Pick  $p$  in one such gap and  $q$  in another.

The BFS ball  $G[L_{<d}]$  is connected, so there exists a simple path  $P$  in  $G[L_{<d}]$  from  $p$  to  $q$ . Define the closed walk

$$W := v \rightarrow p \rightarrow P \rightarrow q \rightarrow v.$$

Every vertex of  $P$  lies in  $L_{<d}$ , while  $\ell(v) = d$ , so  $v$  is distinct from every vertex of  $P$ ;  $P$  is simple, so its internal vertices are distinct; and  $p \neq q$  since they lie in different gaps. Hence  $W$  is a simple cycle in  $G$ .

By the Jordan curve theorem, the planar embedding of  $W$  divides  $\Pi_G$  into two regions. In  $v$ 's rotation, the edges  $v-p$  and  $v-q$  lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by  $W$ . By choice of  $p, q$ , the two arcs of depth- $d$  faces at  $v$  in  $F_{C'}$  lie in different regions of  $W$  (i.e., one arc on each side).

Since  $C'$  is connected in  $G'$  and contains depth- $d$  faces in both arcs, there is a dual path  $f_1, f_2, \dots, f_k$  in  $G'_d$  with  $f_1, f_k \in F_{C'}$  incident to  $v$  in different arcs, and with the intermediate faces  $f_2, \dots, f_{k-1}$  not incident to  $v$  (a shortest such dual path). Consecutive faces  $f_i, f_{i+1}$  share an edge  $e_i$  of  $G$ ; for  $i \geq 2$ , both endpoints of  $e_i$  lie in  $L_{\geq d}$  (since neither  $f_i$  nor  $f_{i+1}$  is incident to  $v$ , all six vertices of these two triangles lie in  $L_{\geq d}$ ). In particular,  $e_i$  shares no endpoint with  $W$  except possibly  $v$  — and  $v$  is excluded from  $f_2, \dots, f_{k-1}$ .

A planar edge with neither endpoint on a simple closed planar curve  $W$  has both of its incident faces on the same side of  $W$ . Applying this to each  $e_i$  ( $i \geq 2$ ) inductively: starting from  $f_2$  on the same side of  $W$  as  $f_1$  (their shared edge  $e_1 = w-w'$  opposite to  $v$  in  $f_1$  has  $w, w' \in L_{\geq d}$  and hence is not on  $W$ ), the path  $f_2 \rightarrow f_3 \rightarrow \dots \rightarrow f_{k-1} \rightarrow f_k$  stays on one side of  $W$ .

But  $f_1$  and  $f_k$  lie on different sides of  $W$  (by construction), contradicting the conclusion that the entire path lies on one side.  $\square$

**Lemma 1.8** (Tire-component lemma). *Let  $G$  be a maximal planar graph and let  $S \subseteq V(G)$  be a level source. Fix a plane embedding  $\Pi_G$  of  $G$  in which  $S$  lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For  $d \geq 0$ , let*

$$G'_d := G'[\{d_f \in V(G') : \delta_G(d_f) = d\}]$$

*be the inner-dual subgraph on dual vertices of dual depth  $d$ , and let  $C'$  be a connected component of  $G'_d$ . Write  $F_{C'} := \{f : d_f \in V(C')\}$ ,  $V_{C'} := \bigcup_{f \in F_{C'}} V(f)$ , and let  $C := G[V_{C'}]$  inherit its embedding from  $\Pi_G$ . Set  $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$ .*

*Then  $C$ , with the inherited embedding, is a tire graph in the sense of Definition 1.5. Its outer boundary  $B_{\text{out}}$  is the side of  $R_{C'}$  closer to  $S$  in  $\Pi_G$ , namely the level- $d$  subgraph  $G[V_{C'} \cap L_d]$  (a simple cycle or single vertex); its inner outerplanar graph is  $O = G[V_{C'} \cap L_{d+1}]$ , and its inner boundary  $B_{\text{in}}$  is the outer-face boundary closed walk of  $O$  in the inherited embedding (a simple cycle when  $O$  is 2-connected,*

a non-simple closed walk in general). The triangular faces of  $C$  inside the closed boundary region are exactly the faces of  $G$  in  $F_{C'}$ .

*Proof. Outerplanarity of the two level parts.* By construction  $S$  lies on the outer face of  $\Pi_G$ , so the outerplanarity lemma of [1] applies directly with  $(G, \Pi_G, S)$ , giving that  $G[L_{d'}]$  is outerplanar for each  $d' \geq 0$ . Subgraphs of outerplanar graphs are outerplanar, so  $G[V_{C'} \cap L_d]$  and  $G[V_{C'} \cap L_{d+1}]$  are both outerplanar.

*Layer containment.* Each  $f \in F_{C'}$  has at least one vertex at level  $d$ , and adjacent vertices in  $G$  differ in level by at most 1; combined with  $\delta_G(d_f) = d$ , this forces  $V(f) \subseteq L_d \cup L_{d+1}$ . Hence  $V_{C'} \subseteq L_d \cup L_{d+1}$ , and  $C$  has vertex partition  $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$ .

*Boundary edges are monochromatic in level.* Each edge  $e$  on  $\partial R_{C'}$  separates a face  $f \in F_{C'}$  from a face  $f' \notin F_{C'}$ . Because  $f$  and  $f'$  share the edge  $e$ , their dual vertices are adjacent in  $G'$ ; if both had depth  $d$  they would lie in the same component of  $G'_d$ , contradicting  $d_f \in C'$  and  $d_{f'} \notin C'$ . Hence  $\delta_G(d_{f'}) \neq d$ ; combined with the bounded-step property of  $\delta$  across  $G'$ -adjacent faces,  $\delta_G(d_{f'}) \in \{d-1, d+1\}$ .

- If  $\delta_G(d_{f'}) = d-1$ , the third vertex  $w$  of  $f' = \{u, v, w\}$  (where  $u, v$  are the endpoints of  $e$ ) has  $\ell(w) = d-1$ . Each of  $u, v$  has  $\ell \in \{d, d+1\}$  (from  $V(f) \subseteq L_d \cup L_{d+1}$ ) and is adjacent to  $w$ , forcing  $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$ .
- If  $\delta_G(d_{f'}) = d+1$ , then all three vertices of  $f'$  lie in  $L_{\geq d+1}$ , so in particular  $\ell(u) = \ell(v) = d+1$ .

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of  $\partial R_{C'}$  is monochromatic in level.

*Boundary structure.* Each connected component of  $\partial R_{C'}$  traces a closed walk in  $G$  that, by the monochromaticity above, lies entirely in  $L_d$  or entirely in  $L_{d+1}$ . By Proposition 1.7, the depth- $d$  faces of  $F_{C'}$  at any  $v \in L_d \cap V_{C'}$  form a single contiguous arc in  $v$ 's rotation, so the source-side boundary walk visits each  $L_d$ -vertex of  $V_{C'}$  exactly once: it is a simple cycle. At vertices  $v \in L_{d+1} \cap V_{C'}$  the depth- $d$  faces may split into multiple arcs of  $v$ 's rotation; this corresponds exactly to  $v$  being a cut-vertex of  $O$ , and the inner-side boundary walk visits  $v$  correspondingly many times — which is already accommodated by Definition 1.5 (where  $B_{\text{in}}$  is the outer-face boundary closed walk of  $O$ , not necessarily a simple cycle).

*Outer boundary.* Because  $S$  lies on the outer face of  $\Pi_G$ , the boundary curve(s) of  $R_{C'}$  on the  $L_d$  side are closer to  $S$  in the embedding. In the inherited embedding of  $C$ , the unique unbounded face is the merged region containing the rest of  $\Pi_G$  outside  $R_{C'}$  on the  $S$  side, so its boundary — a simple cycle on  $L_d$  (or a single vertex when  $V_{C'} \cap L_d = \{v_0\}$ , the  $d = 0$  case) — serves as  $B_{\text{out}}$ . We set  $B_{\text{out}} := G[V_{C'} \cap L_d]$  if this is a cycle, and the single vertex  $\{v_0\}$  in the degenerate case.

*Inner outerplanar graph.* By the outerplanarity lemma of [1],  $G[V_{C'} \cap L_{d+1}]$  is outerplanar. We set  $O := G[V_{C'} \cap L_{d+1}]$ . The boundary curve(s) of  $R_{C'}$  on the  $L_{d+1}$  side are exactly the boundary of  $O$ 's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around  $O$  from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary  $B_{\text{in}}$ . No further restriction on  $O$ 's internal structure is needed: when  $R_{C'}$  has more than two boundary components in the surface-classification sense (i.e. several disjoint simple cycles on  $L_{d+1}$ ), these correspond

precisely to the multiple connected components or bridge crossings of  $O$ , and the outer-face boundary closed walk of  $O$  captures them collectively.

*Tire structure.* The triangular faces of  $C$  inside the closed boundary region are by construction the depth- $d$  faces in  $F_{C'}$ , and the edges of  $C$  are  $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$  where  $E_{\text{ann}}$  are the edges of  $G$  between  $V_{C'} \cap L_d$  and  $V_{C'} \cap L_{d+1}$  that bound a face of  $F_{C'}$ .  $\square$

**Theorem 1.9** (Tire treads partition the bounded faces). *Let  $G$  be a maximal planar graph with planar embedding  $\Pi_G$  and let  $S \subseteq V(G)$  be a level source lying on the outer face. For each  $d \geq 0$  and each connected component  $C'$  of  $G'_d$ , let  $T^{(d,C')}$  denote the tire graph supplied by Lemma 1.8, with tire tread  $R_{C'} \subseteq |\Pi_G|$ . Then the collection of treads*

$$\mathcal{R}(G, S) := \{ R_{C'} : d \geq 0, C' \text{ a connected component of } G'_d \}$$

*partitions the bounded part of  $|\Pi_G|$ :*

- (i) *every bounded face  $f$  of  $G$  is contained in exactly one tread  $R_{C'} \in \mathcal{R}(G, S)$ ;*
- (ii) *distinct treads in  $\mathcal{R}(G, S)$  have disjoint interiors and may share only boundary edges or vertices.*

*Proof. Existence and uniqueness.* Each bounded face  $f \in F(G)$  has a uniquely-defined dual depth  $\delta_G(d_f) \in \mathbb{Z}_{\geq 0}$ , so the dual vertex  $d_f$  lies in  $G'_d$  for  $d = \delta_G(d_f)$  and in no other  $G'_{d'}$ . Within  $G'_d$ , the vertex  $d_f$  belongs to exactly one connected component  $C'$ . By Lemma 1.8,  $F_{C'}$  is precisely the set of faces  $f' \in F(G)$  with  $d_{f'} \in V(C')$ ; in particular  $f \in F_{C'}$ , hence  $f \subseteq R_{C'}$ .

For any other tread  $R_{C''} \in \mathcal{R}(G, S)$ , the component  $C''$  is either at a different depth  $d' \neq d$  (in which case  $F_{C''}$  consists of depth- $d'$  faces and  $f \notin F_{C''}$ ) or at depth  $d$  but a different component  $C'' \neq C'$  (in which case the two components are vertex-disjoint in  $G'_d$ , so again  $f \notin F_{C''}$ ). In both cases  $f \notin R_{C''}$  (more precisely,  $f$  is not one of the triangular faces of  $G$  in  $F_{C''}$ , so  $f$ 's interior is not contained in  $R_{C''}$ ).

*Disjoint interiors.* Each tread  $R_{C'}$  is the union of its triangular faces  $F_{C'} \subseteq F(G)$ ; distinct treads correspond to disjoint  $F_{C'}$  (by the argument above), and the interiors of distinct  $G$ -faces are disjoint. Hence interiors of distinct treads are disjoint.

*Coverage.* Conversely, every bounded face  $f \in F(G)$  has  $d_f \in V(G')$  with some dual depth  $d$ , and thus lies in  $R_{C'}$  where  $C'$  is its component of  $G'_d$ . So  $\bigcup_{R \in \mathcal{R}(G, S)} R$  contains every bounded face of  $G$ .  $\square$

*Remark 1.10.* Either boundary part of  $C$  in Lemma 1.8 may be degenerate. At  $d = 0$  with single-vertex source  $S = \{v_0\}$  the unique component of  $G'_0$  has  $V_{C'} \cap L_0 = \{v_0\}$  as the degenerate *outer* boundary and  $V_{C'} \cap L_1$  a cycle (the link of  $v_0$  in  $G$ ) as the inner boundary. Symmetrically, at  $d = D_{\max}$ ,  $V_{C'} \cap L_{D_{\max}+1} = \emptyset$  degenerates to a single deepest vertex serving as the *inner* boundary, with the level- $D_{\max}$  cycle as the outer boundary.

*Remark 1.11.* Two structural features of  $R_{C'}$  that might at first appear to obstruct the tire-graph conclusion are both already accommodated by Definition 1.5:

*Cut-vertices of  $O$ .* A vertex  $v \in V_{C'} \cap L_{d+1}$  may have the faces of  $F_{C'}$  incident to it split into two or more arcs in  $v$ 's rotation in  $\Pi_G$ , separated by faces of higher depth. This corresponds exactly to  $v$  being a cut-vertex of  $O = G[V_{C'} \cap L_{d+1}]$ , and

the inner boundary closed walk  $B_{\text{in}}$  then visits  $v$  multiple times — once for each arc. No additional hypothesis is needed.

*Multi-hole topology of  $R_{C'}$ .* Even when  $R_{C'}$  encloses several disjoint depth- $> d$  sub-regions, the inner outerplanar graph  $O$  captures the multi-hole structure as a disconnected or non-2-connected outerplanar graph (with bridges or multiple components), and its outer-face boundary closed walk serves as  $B_{\text{in}}$  traversing bridges twice and visiting cut-vertices multiple times.

In the special case  $d = 0$  with single-vertex source  $S = \{v_0\}$ ,  $R_{C'}$  is the star of  $v_0$ , a topological closed disk with one boundary cycle (the link of  $v_0$ ); the corresponding tire graph has degenerate outer boundary  $\{v_0\}$ .

**Theorem 1.12** (Inner dual of a tire tread is outerplanar). *Let  $T = (B_{\text{out}}, O, E_{\text{ann}})$  be a tire graph, and let  $\Gamma$  be the graph on vertex set  $\{d_f : f \in F_{\text{ann}}\}$  with an edge  $d_f d_{f'}$  for each interior annular edge of  $T$  (= each edge of  $T$  whose two incident faces both lie in  $F_{\text{ann}}$ ). Then  $\Gamma$  is outerplanar.*

*Moreover,  $\Gamma$  admits a planar embedding as a (possibly non-simple) Hamilton walk through every  $d_f$ , plus zero or more non-crossing chords.*

*Proof.* We argue by cases on whether the tire tread  $R$  is a disk or an annulus.

*Case 1:  $R$  is a closed disk* (one of  $B_{\text{out}}, B_{\text{in}}$  degenerate, by Definition 1.5). Then  $T \cap R$  triangulates a polygon with no interior vertices: the polygon's boundary cycles round the unique non-degenerate boundary plus the degenerate apex, and all vertices of  $V(T) \cap R$  lie on this polygon. The dual graph of such a polygon triangulation is a tree (a classical fact: a triangulation of a  $p$ -gon with no interior vertex has  $p - 2$  triangles and  $p - 3$  diagonals, and the diagonals' adjacency graph is a tree). Trees are outerplanar.

*Case 2:  $R$  is an annulus* (both  $B_{\text{out}}$  and  $B_{\text{in}}$  non-degenerate). We construct an explicit outerplanar embedding of  $\Gamma$  as a Hamilton walk plus non-crossing chords.

*Step 1: Cyclic ordering of  $F_{\text{ann}}$ .* The boundary of the annular tread is the disjoint union  $\partial R = B_{\text{out}} \sqcup \overline{B_{\text{in}}}$  (viewing  $B_{\text{in}}$  as a closed walk traced in the appropriate orientation). Each boundary edge of  $R$  is incident to exactly one annular face: walking around  $B_{\text{out}}$  in cyclic order produces a sequence  $f_1^{\text{out}}, f_2^{\text{out}}, \dots, f_n^{\text{out}}$  of (not necessarily distinct) annular faces, one per  $B_{\text{out}}$ -edge; similarly walking around  $B_{\text{in}}$  produces a sequence  $f_1^{\text{in}}, \dots, f_{m_\partial}^{\text{in}}$  where  $m_\partial$  is the length of the inner-boundary walk. Pick any spoke  $e^* = uw \in E_{\text{ann}}$  with  $u \in V(B_{\text{out}})$  and  $w \in V(B_{\text{in}})$ ; cut  $R$  along  $e^*$ . This converts the annulus into a closed disk  $\tilde{R}$  whose boundary walks once around  $B_{\text{out}}$ , once along  $e^*$ , once around  $B_{\text{in}}$  in reverse, and once back along  $e^*$ . Concatenating the two boundary sequences (in the order dictated by this disk traversal) yields a single cyclic sequence

$$\mathcal{S} = (f_1^{\text{out}}, \dots, f_n^{\text{out}}, f_1^{\text{in}}, \dots, f_{m_\partial}^{\text{in}})$$

of annular faces with multiplicities.

*Step 2: The Hamilton walk.* Consecutive entries of  $\mathcal{S}$  correspond either to the same annular face (when two adjacent boundary edges meet at a vertex incident to a single annular face) or to two annular faces sharing an interior edge of  $E_{\text{ann}}$ . In the former case the walk stays at one  $\Gamma$ -vertex; in the latter it uses one  $\Gamma$ -edge. The resulting closed walk in  $\Gamma$  visits every face that appears in  $\mathcal{S}$  at least once.

If every  $f \in F_{\text{ann}}$  appears in  $\mathcal{S}$  (i.e. every annular face has at least one boundary edge of  $R$ ), the walk is a Hamilton walk in  $\Gamma$ , and we are done up to Step 3. Each annular face with two boundary edges contributes a vertex visited twice; each with three contributes a vertex visited three times.

If some  $f \in F_{\text{ann}}$  does not appear in  $\mathcal{S}$  (i.e. has no boundary edge of  $R$ ), then all three edges of  $f$  are interior annular edges, so  $d_f$  has degree 3 in  $\Gamma$ . Such a face is “trapped” in the interior of the dual graph and appears as the endpoint of a chord. Extend the walk by: whenever it crosses an interior annular edge  $e$  shared with a boundary-free face  $f$ , detour through  $f$  and back. After finitely many such detours (one per boundary-free face), the walk becomes a Hamilton walk visiting every  $d_f$ .

*Step 3: Non-crossing chords.* The  $\Gamma$ -edges not used by the Hamilton walk constructed in Step 2 are the remaining interior annular edges. Each such edge  $e \in E_{\text{ann}}$  corresponds to a chord between two non-adjacent positions of  $\mathcal{S}$ . In the inherited planar embedding of  $\Gamma$  in  $R$ , these chords are drawn as straight segments between annular triangle centroids; *they do not cross* because the underlying  $E_{\text{ann}}$  edges they cross are themselves non-crossing in the planar embedding of  $T$ .

*Step 4: Outerplanar embedding.* We now lay out  $\Gamma$  as follows: place the  $|F_{\text{ann}}|$  vertices on a circle in the cyclic order given by  $\mathcal{S}$  (treating multiply-visited faces as single circle vertices). Connect consecutive vertices on the circle by the Hamilton-walk edges, which forms the closed walk. Draw the remaining edges as chords inside the circle. Because the chords were non-crossing in  $T$ ’s planar embedding, they remain non-crossing here. All vertices lie on the outer face (the unbounded region outside the circle), making  $\Gamma$  outerplanar.  $\square$

*Remark 1.13.* In the *spoke-only* case (Definition 1.5 with  $O$  2-connected and  $E_{\text{ann}}$  consisting only of spokes), every annular face has exactly one boundary edge, every  $d_f$  has  $\Gamma$ -degree 2, and the construction of the Theorem 1.12 proof reduces to the classical Hamilton cycle  $\Gamma \cong C_{n+m}$  with zero chords.

*Remark 1.14.* When  $O$  has a bridge  $e_{\text{br}} \in E(O)$  whose two incident faces are annular triangles,  $e_{\text{br}}$  contributes an interior annular edge in  $\Gamma$  rather than two leaves in  $D(T)$  (see Definition 1.7 of [2]). The two bridge-incident annular triangles have  $\Gamma$ -degree 3; the resulting  $\Gamma$  has the structure of a Hamilton cycle of length  $n + m_{\partial}$  plus a single chord (length 1). This corresponds to the theta graph  $\Theta(1, b, c)$  identified empirically in [2], which has no  $K_{2,3}$  subdivision (since one of the three paths has length 1 and so contributes no degree-2 branch vertex), hence is outerplanar as predicted.

## REFERENCES

- [1] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.
- [2] E. Bauerfeld, *Coloring Nested Tire Dual Graphs*, manuscript (math-research repository), 2026.